E-Values

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July 1, 2026

Chapter 1

Extended real numbers and their integral

TODO: describe the extended real numbers and the conventions used for their operations.

Definition 1.1. The set of extended non-negative real numbers is $\mathbb{R}_{+,\infty} = \mathbb{R}_+ \cup \{+\infty\}$.

Definition 1.2. The set of extended real numbers is $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

TODO: addition and multiplication conventions

Definition 1.3. The logarithm as a function from $\mathbb{R}_{+,\infty}$ to $\overline{\mathbb{R}}$ is defined as

$$\log(x) = \begin{cases} -\infty & \text{if } x = 0 \\ \text{the usual logarithm} & \text{if } x \in (0, +\infty) \\ +\infty & \text{if } x = +\infty \end{cases}$$

A good property of the conventions chosen for addition and multiplication is that the usual properties of logarithm hold. In particular, for all $x, y \in \mathbb{R}_{+,\infty}$, $\log(xy) = \log(x) + \log(y)$.

TODO: positive and negative parts of an extended real number

Definition 1.4. For $f : \Omega \to \overline{\mathbb{R}}$ and μ a measure on Ω, we define the extended integral of f with respect to μ as

$$\int_\omega f(\omega)\, d\mu := \int_\omega f^+(\omega)\, d\mu - \int_\omega f^-(\omega)\, d\mu \,,$$

where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, and their integrals are Lebesgue integrals, possibly taking the value $+\infty$.

Chapter 2

E-variables

Notation Let $\mathcal{X}, \mathcal{Y}$ be measurable spaces. We denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$ and write $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ to mean that κ is a Markov kernel from $\mathcal{X}$ to $\mathcal{Y}$. $\mathbb{R}_{+,\infty}$ is the set of nonnegative reals extended with $+\infty$ and $\overline{\mathbb{R}}$ is the set of reals extended with $-\infty$ and $+\infty$.

A measure will always denote a nonnegative measure. We will specify when we mean a signed measure.

The central object of this study are E-variables, which are measurable functions that are bounded in expectation by 1 for all probability measures in a set $\mathcal{Q}$.

Definition 2.1. An E-variable for a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ is a measurable function $f : \mathcal{X} \to \mathbb{R}_{+,\infty}$ such that for all $Q \in \mathcal{Q}$, $Q[f] \le 1$.

We denote by $\mathcal{E}_{\mathcal{Q}}$ the set of E-variables for a set of probability measures $\mathcal{Q}$.

Lemma 2.2. *If $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ then $\mathcal{E}_{\mathcal{Q}} \subseteq \mathcal{E}_{\mathcal{P}}$.*

Proof. □

We introduce randomized E-variables, which are Markov kernels that can be seen as randomized versions of E-variables. The main difference is that they are allowed to be randomized, i.e., they can return a distribution over $\mathbb{R}_{+,\infty}$ instead of a single value. This generalization is mainly useful thanks to the compositional properties it provides, as we will see in Lemma 2.22.

Definition 2.3. A randomized E-variable for a set of distributions $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ is a Markov kernel $\eta : \mathcal{X} \rightsquigarrow \mathbb{R}_{+,\infty}$ such that for all $Q \in \mathcal{Q}$, $(\eta \circ Q)[\mathrm{id}] \le 1$.

We denote by $\mathcal{E}^R_{\mathcal{Q}}$ the set of randomized E-variables for a set of probability measures $\mathcal{Q}$. Since measurable functions can be seen as deterministic Markov kernels, we have $\mathcal{E}_{\mathcal{Q}} \subseteq \mathcal{E}^R_{\mathcal{Q}}$. Indeed, $Q[f] = (\delta_f \circ Q)[\mathrm{id}]$, in which $\delta_f : \mathcal{X} \rightsquigarrow \mathbb{R}_{+,\infty}$ is such that $\delta_f(x)$ is the dirac distribution at $f(x)$.

Lemma 2.4. *A Markov kernel $\eta : \mathcal{X} \rightsquigarrow \mathbb{R}_{+,\infty}$ is a randomized E-variable for a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ if and only if the measurable function $f : \mathcal{X} \to \mathbb{R}_{+,\infty}$ defined by $f(x) = \eta(x)[\mathrm{id}]$ is an E-variable for $\mathcal{Q}$.*

Proof. □

2.1 Almost everywhere with respect to a set of measures

Definition 2.5. Let $\mathcal{Q} \subseteq \mathcal{M}(\mathcal{X})$ be a set of measures. A set $S \subseteq \mathcal{X}$ is said to be $\mathcal{Q}$-null if $Q[S] = 0$ for all $Q \in \mathcal{Q}$. A property $p : \mathcal{X} \to \mathrm{Bool}$ is said to hold $\mathcal{Q}$-almost everywhere (denoted by $\mathcal{Q}$-a.e.) if the set $\{x \mid \neg p(x)\}$ is $\mathcal{Q}$-null.

For a measure $P \in \mathcal{M}(\mathcal{X})$, we can define the almost everywhere filter ae_P of sets with P-null complement.

Lemma 2.6. *A set $S \subseteq \mathcal{X}$ has a $\mathcal{Q}$-null complement iff $S \in \bigsqcup_{Q \in \mathcal{Q}} \mathrm{ae}_Q$.*

Proof. By definition of the ae filter and of a supremum of filters. □

Definition 2.7. Let $P \in \mathcal{M}(\mathcal{X})$ and let $\mathcal{Q} \subseteq \mathcal{M}(\mathcal{X})$ be a set of measures. We say that P is absolutely continuous with respect to $\mathcal{Q}$ (denoted by $P \ll \mathcal{Q}$) if all $\mathcal{Q}$-null sets are P-null sets.

Lemma 2.8. *Let $P \in \mathcal{M}(\mathcal{X})$ and let $\mathcal{Q} \subseteq \mathcal{M}(\mathcal{X})$. Then $P \ll \mathcal{Q}$ iff $\mathrm{ae}_P \le \bigsqcup_{Q \in \mathcal{Q}} \mathrm{ae}_Q$.*

Proof. □

Lemma 2.9. *For $P \in \mathcal{M}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{M}(\mathcal{X})$, there is a unique decomposition $P = P_{\mathrm{ac}} + P_{\perp}$ with $P_{\mathrm{ac}} \ll \mathcal{Q}$ and $P_{\perp} \perp \mathcal{Q}$ (meaning that there exists a $\mathcal{Q}$-null set A with $P_{\perp}(A^c) = 0$).*

Proof. □

Definition 2.10. The singular set of P with respect to $\mathcal{Q}$ is a $\mathcal{Q}$-null set A such that the singular part $P_{\perp}$ satisfies $P_{\perp}(A^c) = 0$.

Lemma 2.11. *The singular set of P with respect to $\mathcal{Q}$ is unique up to P and $\mathcal{Q}$-null sets.*

Proof. □

Lemma 2.12. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and let f be an E-variable for $\mathcal{Q}$. Then the set $\{f = \infty\}$ is $\mathcal{Q}$-null.*

Proof. Let $Q \in \mathcal{Q}$. Since f is an E-variable for $\mathcal{Q}$, we have $Q[f] \le 1$. Hence $Q[\{f = \infty\}] = 0$. □

2.2 Utility and maximal utility

Definition 2.13. We call utility a function $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ which is non-decreasing, concave, finite except perhaps at 0 and ∞, continuous, and continuously differentiable on $(0, \infty)$.

Definition 2.14. The logarithm satisfies the properties of a utility function and we call it logarithmic utility.

Definition 2.15. For a utility function U, the maximal utility for $(P, \mathcal{Q})$ is defined as

$$\mathcal{U}(P, \mathcal{Q}) = \sup_{f \in \mathcal{E}_{\mathcal{Q}}} P[U(f)] \, ,$$

in which the expectation is the extended integral defined in Definition 1.4.

Definition 2.16. For a utility function U, the maximal randomized utility for $(P, \mathcal{Q})$ is defined as

$$\mathcal{U}^R(P, \mathcal{Q}) = \sup_{\eta \in \mathcal{E}^R_{\mathcal{Q}}} (\eta \circ P)[U] ,$$

in which the expectation is the extended integral defined in Definition 1.4.

Lemma 2.17. *Let $\mathcal{Q}' \subseteq \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. Then*

$$\mathcal{U}(P, \mathcal{Q}') \geq \mathcal{U}(P, \mathcal{Q}) .$$

Proof. ◻

The next lemma shows that the larger set of randomized E-variables does not allow to achieve a larger expected utility than the set of E-variables.

Lemma 2.18. *Let $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ be a utility function and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ a set of probability measures. Let $P \in \mathcal{P}(\mathcal{X})$. Then*

$$\mathcal{U}^R(P, \mathcal{Q}) = \mathcal{U}(P, \mathcal{Q}) .$$

Proof. Since $\mathcal{E}_{\mathcal{Q}} \subseteq \mathcal{E}^R_{\mathcal{Q}}$, the left hand side is greater than or equal to the right hand side.

For the other direction, let $\eta \in \mathcal{E}^R_{\mathcal{Q}}$ be a randomized E-variable. By concavity of U,

$$(\eta \circ P)[U] = P[x \mapsto \eta(x)[U]] \leq P[x \mapsto U(\eta(x)[\mathrm{id}])]$$

The function $x \mapsto \eta(x)[\mathrm{id}]$ is a measurable function and belongs to $\mathcal{E}_{\mathcal{Q}}$. ◻

Lemma 2.19. *Suppose that there exists an event A such that $P(A) > 0$ and $Q(A) = 0$ for all $Q \in \mathcal{Q}$. That is, we don't have $P \ll \mathcal{Q}$. Then*

$$\mathcal{U}(P, \mathcal{Q}) = +\infty .$$

Proof. The function $f : \mathcal{X} \to \mathbb{R}_{+,\infty}$ defined by $f(x) = +\infty$ if $x \in A$ and $f(x) = 0$ otherwise is an E-variable for $\mathcal{Q}$ since $Q[f] = 0$ for all $Q \in \mathcal{Q}$.

$$\mathcal{U}(P, \mathcal{Q}) \geq P[\log f] = P(A) \cdot (+\infty) = +\infty .$$

◻

Lemma 2.20. *The function $R \mapsto \mathcal{U}(R, \mathcal{P})$ is convex.*

Proof. Let $R_1, R_2 \in \mathcal{P}(\mathcal{X})$ and let $\theta \in [0, 1]$.

$$\begin{aligned}
\mathcal{U}(\theta R_1 + (1-\theta)R_2, \mathcal{P}) &= \sup_{f \in \mathcal{E}_{\mathcal{P}}} (\theta R_1 + (1-\theta)R_2)\,[\log f] \\
&= \sup_{f \in \mathcal{E}_{\mathcal{P}}} \{\theta R_1[\log f] + (1-\theta) R_2[\log f]\} \\
&\leq \theta \sup_{f \in \mathcal{E}_{\mathcal{P}}} R_1[\log f] + (1-\theta) \sup_{f \in \mathcal{E}_{\mathcal{P}}} R_2[\log f] \\
&= \theta \mathcal{U}(R_1, \mathcal{P}) + (1-\theta)\mathcal{U}(R_2, \mathcal{P}) .
\end{aligned}$$

◻

2.3 Data-processing inequality for e-variables

Lemma 2.21. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and let $\phi : \mathcal{X} \to \mathcal{Y}$ be a measurable function. Then for all $g \in \mathcal{E}_{\phi_* \mathcal{Q}}$, $g \circ \phi \in \mathcal{E}_{\mathcal{Q}}$.*

Proof. For $Q \in \mathcal{Q}$,

$$Q[g \circ \phi] = (\phi_* Q)[g] \le 1 .$$

□

Lemma 2.22. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel. Then for all $\xi \in \mathcal{E}^R_{\kappa \circ \mathcal{Q}}$, $\xi \circ \kappa \in \mathcal{E}^R_{\mathcal{Q}}$.*

Proof. For $Q \in \mathcal{Q}$,

$$((\xi \circ \kappa) \circ Q)[\mathrm{id}] = (\xi \circ (\kappa \circ Q))[\mathrm{id}] \le 1 .$$

□

Theorem 2.23 (Data processing inequality with randomized e-variables)**.** *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and $P \in \mathcal{P}(\mathcal{X})$. Let $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ be a utility function. Then for all Markov kernels $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$,*

$$\mathcal{U}^R(P, \mathcal{Q}) \ge \mathcal{U}^R(\kappa \circ P, \kappa \circ \mathcal{Q}) .$$

Proof. By Lemma 2.22, $\{\xi \circ \kappa \mid \xi \in \mathcal{E}^R_{\kappa \circ \mathcal{Q}}\} \subseteq \mathcal{E}^R_{\mathcal{Q}}$, hence

$$\sup_{\eta \in \mathcal{E}^R_{\mathcal{Q}}} (\eta \circ P)[U] \ge \sup_{\eta \in \{\xi \circ \kappa \mid \xi \in \mathcal{E}^R_{\kappa \circ \mathcal{Q}}\}} (\eta \circ P)[U] = \sup_{\xi \in \mathcal{E}^R_{\kappa \circ \mathcal{Q}}} (\xi \circ \kappa \circ P)[U] .$$

□

Lemma 2.24 (Data processing inequality with e-variables)**.** *Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel, $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. Then*

$$\mathcal{U}(\kappa \circ P, \kappa \circ \mathcal{Q}) \le \mathcal{U}(P, \mathcal{Q}) .$$

Proof. We can use Lemma 2.18 to obtain the data-processing with a Markov kernel for e-variables from the result for randomized e-variables (Theorem 2.23). □

Lemma 2.25 (Data processing inequality with e-variables)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a measurable function, $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. Then*

$$\mathcal{U}(f_* P, f_* \mathcal{Q}) \le \mathcal{U}(P, \mathcal{Q}) .$$

Proof. Use Lemma 2.24 with the deterministic Markov kernel given by f. □

Lemma 2.26. *Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a measurable embedding, $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. Then*

$$\mathcal{U}(\phi_* P, \phi_* \mathcal{Q}) = \mathcal{U}(P, \mathcal{Q}) .$$

Proof.

□

Lemma 2.27. *Let $R \in \mathcal{P}(\mathcal{X})$, $\mathcal{P} \subseteq \mathcal{P}(\mathcal{X})$ and let $\phi : \mathcal{X} \to \mathcal{X}$ be a measurable map such that $\phi \circ \phi = \mathrm{id}$. Then*

$$\mathcal{U}(R, \phi_* \mathcal{P}) = \mathcal{U}(\phi_* R, \mathcal{P}) .$$

Proof. To prove the first inequality, we use Lemma 2.25:

$$\begin{aligned} \mathcal{U}(R, \phi_* \mathcal{P}) &= \mathcal{U}(\phi_*(\phi_* R), \phi_* \mathcal{P}) \\ &\leq \mathcal{U}(\phi_* R, \mathcal{P}) . \end{aligned}$$

For the reverse inequality,

$$\begin{aligned} \mathcal{U}(\phi_* R, \mathcal{P}) &= \mathcal{U}(\phi_* R, \phi_*(\phi_* \mathcal{P})) \\ &\leq \mathcal{U}(R, \phi_* \mathcal{P}) . \end{aligned}$$

□

2.4 Numeraire and duality

This section gathers results from [LRR24].

Definition 2.28. Let $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. A numeraire for $(P, \mathcal{Q})$ is a P-almost surely positive e-variable $f^* \in \mathcal{E}_\mathcal{Q}$ such that $P[f/f^*] \leq 1$ for all $f \in \mathcal{E}_\mathcal{Q}$.

Lemma 2.29. *Let f^* be a numeraire for $(P, \mathcal{Q})$ and let $f \in \mathcal{E}_\mathcal{Q}$. Then $P\left[\log \frac{f}{f^*}\right] \leq 0$.*

Proof. □

Lemma 2.30. *Let f be a P-almost surely positive E-variable for a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. Then f is a numeraire for $(P, \mathcal{Q})$ if and only if $P[\log \frac{g}{f}] \leq 0$ for all $g \in \mathcal{E}_\mathcal{Q}$.*

Proof.

□

Lemma 2.31. *Let f^* be a numeraire for $(P, \mathcal{Q})$. Then for all $f \in \mathcal{E}_\mathcal{Q}$,*

$$P[\log f] \leq P[\log f^*] .$$

Proof.

□

Lemma 2.32. *Let f^* be a numeraire for $(P, \mathcal{Q})$. Then $P[\log f^*] \geq 0$.*

Proof. Use Lemma 2.31 with the e-variable $f = 1$. □

Lemma 2.33. *Let $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$. If f^* is a numeraire for $(P, \mathcal{Q}_1)$ and $f^* \in \mathcal{E}_{\mathcal{Q}_2}$ then f^* is also a numeraire for $(P, \mathcal{Q}_2)$.*

Proof. For $f \in \mathcal{E}_{\mathcal{Q}_2}$, we have $f \in \mathcal{E}_{\mathcal{Q}_1}$ since $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$. Then $P[f/f^*] \leq 1$ by the numeraire property for $(P, \mathcal{Q}_1)$. Thus f^* is a numeraire for $(P, \mathcal{Q}_2)$. □

Lemma 2.34. *Let $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. If f^* is a numeraire for $(P, \mathcal{Q})$ then*

$$\mathcal{U}(P, \mathcal{Q}) = P[U(f^*)] .$$

Proof. □

2.4.1 Existence of a numeraire

Lemma 2.35. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions from $\mathcal{X}$ to $\mathbb{R}_{+,\infty}$ and let $Q \in \mathcal{P}(\mathcal{X})$. There exists a sequence of measurable functions $\tilde{X}_n \in \mathrm{conv}(X_n, X_{n+1}, \ldots)$ and $X : \mathcal{X} \to \mathbb{R}_{+,\infty}$ such that $\tilde{X}_n \to X$ Q-almost surely.*

Proof. Proved in the Brownian motion project. □

Lemma 2.36. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ a concave function which is bounded from above. Then there exists $f^* \in \mathcal{E}_{\mathcal{Q}}$ such that*

$$\sup_{f\in\mathcal{E}_{\mathcal{Q}}} P[U(f)] = P[U(f^*)] \, .$$

The properties of $\mathcal{E}_{\mathcal{Q}}$ we use for the proof of this lemma are convexity and the fact that if $f_n \in \mathcal{E}_{\mathcal{Q}}$ then $\liminf_{n\to\infty} f_n \in \mathcal{E}_{\mathcal{Q}}$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of e-variables in $\mathcal{E}_{\mathcal{Q}}$ such that $n \mapsto P[U(f_n)]$ is non-decreasing and $P[U(f_n)] \to \sup_{f\in\mathcal{E}_{\mathcal{Q}}} P[U(f)]$. Let $\tilde{f}_n$ be a sequence of measurable functions and let $\tilde{f} : \mathcal{X} \to \mathbb{R}_{+,\infty}$, with $\tilde{f}_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$ such that $\tilde{f}_n \to \tilde{f}$ P-almost surely. That sequence exists by Lemma 2.35. By convexity of $\mathcal{E}_{\mathcal{Q}}$, $\tilde{f}_n \in \mathcal{E}_{\mathcal{Q}}$ for all n.

Let $f^* = \liminf_{n\to\infty} \tilde{f}_n$. Note that f^* is a measurable function such that $f^* = \tilde{f}$ P-almost surely. We have $f^* \in \mathcal{E}_{\mathcal{Q}}$ by an application of Fatou's lemma: for $Q \in \mathcal{Q}$,

$$Q[f^*] = Q[\liminf_{n\to\infty} \tilde{f}_n] \le \liminf_{n\to\infty} Q[\tilde{f}_n] \le 1 \, .$$

By concavity of U and the fact that $P[U(f_n)]$ is non-decreasing, we have

$$P[U(\tilde{f}_n)] = P[U(\sum_{i=n}^{m_n} \lambda_{n,i} f_i)] \ge \sum_{i=n}^{m_n} \lambda_{n,i} P[U(f_i)] \ge P[U(f_n)] \, .$$

We also have $f^* = \limsup_{n\to\infty} \tilde{f}_n$ P-almost surely, hence by Fatou's lemma (using that U is bounded from above),

$$P[U(f^*)] = P[U(\limsup_{n\to\infty} \tilde{f}_n)] \ge \limsup_{n\to\infty} P[U(\tilde{f}_n)] \ge \limsup_{n\to\infty} P[U(f_n)] = \sup_{f\in\mathcal{E}_{\mathcal{Q}}} P[U(f)] \, .$$

Since $f^* \in \mathcal{E}_{\mathcal{Q}}$, we also have $P[U(f^*)] \le \sup_{f\in\mathcal{E}_{\mathcal{Q}}} P[U(f)]$, which proves the equality. □

Lemma 2.37. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ a concave function which is bounded from above. Then there exists $f^* \in \mathcal{E}_{\mathcal{Q}}$ such that*

$$\sup_{f\in\mathcal{E}_{\mathcal{Q}}} P[U(f)] = P[U(f^*)] \, ,$$

and such that all e-variables are P-a.s. finite on $\{f^* < \infty\}$.

Proof. Let f_0 be as in Lemma 2.36. The set $\{f_0 = \infty\}$ is $\mathcal{Q}$-null by Lemma 2.12.

TODO □

Lemma 2.38. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures and $U : \mathbb{R}_{+,\infty} \to \overline{\mathbb{R}}$ a non-decreasing concave C^1 function which is bounded from above. Then the e-variable f^* in Lemma 2.36 is such that for all e-variables $g \in \mathcal{E}_{\mathcal{Q}}$,*

$$P\left[U'(f^*)(g - f^*)\right] \le 0\,.$$

Proof. For $f \in \mathcal{E}_{\mathcal{Q}}$, let $f_t = tf + (1-t)f^*$ for $t \in [0,1]$. By concavity of U, $U(f_t) \ge tU(f) + (1-t)U(f^*)$, hence

$$\frac{U(f_t) - U(f^*)}{t} \ge U(f) - U(f^*)\,.$$

TODO □

Theorem 2.39 (Existence of the numeraire)**.** *Let $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. A numeraire for $(P, \mathcal{Q})$ exists and is P-a.e. unique.*

Proof.

□

We will write $E^*_{P,\mathcal{Q}}$ for the numeraire for $(P, \mathcal{Q})$.

2.4.2 Products

Theorem 2.40 (Numeraire of a product)**.** *Let $P_1 \in \mathcal{P}(\mathcal{X}), P_2 \in \mathcal{P}(\mathcal{Y})$ and $\mathcal{Q}_1 \subseteq \mathcal{P}(\mathcal{X}), \mathcal{Q}_2 \subseteq \mathcal{P}(\mathcal{Y})$. Then the numeraire for the product $(P_1 \times P_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ is given by the product of the numeraires for $(P_1, \mathcal{Q}_1)$ and $(P_2, \mathcal{Q}_2)$:*

$$E^*_{P_1 \times P_2, \mathcal{Q}_1 \times \mathcal{Q}_2} = E^*_{P_1, \mathcal{Q}_1} E^*_{P_2, \mathcal{Q}_2}\,.$$

Its logarithmic utility is given by

$$(P_1 \times P_2)[\log E^*_{P_1 \times P_2, \mathcal{Q}_1 \times \mathcal{Q}_2}] = P_1[\log E^*_{P_1, \mathcal{Q}_1}] + P_2[\log E^*_{P_2, \mathcal{Q}_2}]\,.$$

Proof. We first need to check that $E^*_{P_1, \mathcal{Q}_1} E^*_{P_2, \mathcal{Q}_2}$ is in $\mathcal{E}_{\mathcal{Q}_1 \times \mathcal{Q}_2}$. For $Q_1 \in \mathcal{Q}_1$ and $Q_2 \in \mathcal{Q}_2$, we have

$$(Q_1 \times Q_2)\left[E^*_{P_1, \mathcal{Q}_1} E^*_{P_2, \mathcal{Q}_2}\right] = Q_1[E^*_{P_1, \mathcal{Q}_1}] Q_2[E^*_{P_2, \mathcal{Q}_2}] \le 1\,.$$

Let's now check the numeraire property. Let $f \in \mathcal{E}_{\mathcal{Q}_1 \times \mathcal{Q}_2}$. Let $E_1^* = E^*_{P_1, \mathcal{Q}_1}$ and $E_2^* = E^*_{P_2, \mathcal{Q}_2}$.

$$(P_1 \times P_2)\left[\frac{f}{E_1^* E_2^*}\right] = P_1\left[\frac{1}{E_1^*} P_2\left[\frac{f}{E_2^*}\right]\right]\,.$$

If we can show that $x \mapsto P_2\left[\frac{f(x,\cdot)}{E_2^*}\right]$ is in $\mathcal{E}_{\mathcal{Q}_1}$, then we can apply the definition of the numeraire for $(P_1, \mathcal{Q}_1)$ to obtain

$$(P_1 \times P_2)\left[\frac{f}{E_1^* E_2^*}\right] = P_1\left[\frac{1}{E_1^*} P_2\left[\frac{f}{E_2^*}\right]\right] \le 1\,.$$

Let then $Q_1 \in \mathcal{Q}_1$. We want to show that $Q_1\left[P_2\left[\frac{f(x,\cdot)}{E_2^*}\right]\right] \le 1$. By Fubini's theorem, we have

$$Q_1\left[P_2\left[\frac{f(x,\cdot)}{E_2^*}\right]\right] = P_2\left[\frac{Q_1\left[f(\cdot, y)\right]}{E_2^*}\right]\,.$$

It now suffices to show that $y \mapsto Q_1[f(\cdot, y)]$ is in $\mathcal{E}_{\mathcal{Q}_2}$. Let $Q_2 \in \mathcal{Q}_2$. Then

$$Q_2[y \mapsto Q_1[f(\cdot, y)]] = (Q_1 \times Q_2)[f] \leq 1 \, .$$

The last inequality holds since $f \in \mathcal{E}_{\mathcal{Q}_1 \times \mathcal{Q}_2}$. ☐

Lemma 2.41. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ and $H \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})$ be a set of Markov kernels. If $f \in \mathcal{E}_{\mathcal{Q} \otimes H}$ then for all $\eta \in H$, the function $x \mapsto \eta(x)[f(x, \cdot)]$ is in $\mathcal{E}_{\mathcal{Q}}$.*

Proof. For $Q \in \mathcal{Q}$ and $f \in \mathcal{E}_{\mathcal{Q} \otimes H}$, $Q[x \mapsto \eta(x)[f]] = (Q \otimes \eta)[f] \leq 1$, hence $(x \mapsto \eta(x)[f(x, \cdot)]) \in \mathcal{E}_{\mathcal{Q}}$. ☐

Lemma 2.42. *$E^*_{P \otimes \kappa, \mathcal{Q} \otimes \kappa} =_{(P \otimes \kappa) a.e.} (x, y) \mapsto E^*_{P, \mathcal{Q}}(x)$ for any Markov kernel $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$.*

Proof. First we need to show that $E^*_{P,\mathcal{Q}}$ is in $\mathcal{E}_{\mathcal{Q} \otimes \kappa}$. It is measurable and for $Q \in \mathcal{Q}$,

$$(Q \otimes \kappa)\left[E^*_{P,\mathcal{Q}}\right] = Q\left[E^*_{P,\mathcal{Q}}\right] \leq 1 \, .$$

We then check the numeraire condition. By Lemma 2.41, $(x \mapsto \kappa(x)[f]) \in \mathcal{E}_{\mathcal{Q}}$. By the definition of the numeraire, we then have

$$P\left[\frac{\kappa(x)[f]}{E^*_{P,\mathcal{Q}}}\right] \leq 1 \, .$$

That is, $(P \otimes \kappa)\left[\frac{f}{E^*_{P,\mathcal{Q}}}\right] \leq 1$ for all $f \in \mathcal{E}_{\mathcal{Q} \otimes \kappa}$. We have proved that $E^*_{P,\mathcal{Q}}$ is the numeraire for $(P \otimes \kappa, \mathcal{Q} \otimes \kappa)$. ☐

Theorem 2.43. *For any $f \in \mathcal{E}_{\mathcal{Q} \otimes H}$, we have*

$$(P \otimes \kappa)\left[\frac{f}{E^*_{P,\mathcal{Q}} E^*_{P \otimes \kappa, \{P\} \otimes H}}\right] \leq 1 \, .$$

Note that this proves the numeraire property for the candidate $E^*_{P,\mathcal{Q}} E^*_{P \otimes \kappa, P \otimes H}$, but we don't know if it is an e-variable, so can't conclude that it's the numeraire. As a consequence,

$$(P \otimes \kappa)\left[\log E^*_{P \otimes \kappa, \mathcal{Q} \otimes H}\right] \leq P\left[\log E^*_{P,\mathcal{Q}}\right] + (P \otimes \kappa)\left[\log E^*_{P \otimes \kappa, \{P\} \otimes H}\right] \, .$$

If $E^*_{P,\mathcal{Q}} E^*_{P \otimes \kappa, \{P\} \otimes H}$ was an e-variable, then we would have equality.

Proof. Let $f \in \mathcal{E}_{\mathcal{Q} \otimes H}$.

$$(P \otimes \kappa)\left[\frac{f}{E^*_{P,\mathcal{Q}} E^*_{P \otimes \kappa, P \otimes H}}\right] = (P \otimes \kappa)\left[\frac{f / E^*_{P,\mathcal{Q}}}{E^*_{P \otimes \kappa, P \otimes H}}\right] \, .$$

It suffices to show that $f / E^*_{P,\mathcal{Q}}$ is in $\mathcal{E}_{P \otimes H}$ to obtain that this is less than 1 by the numeraire property. Let $\eta \in H$. By Lemma 2.42,

$$(P \otimes \eta)\left[\frac{f}{E^*_{P,\mathcal{Q}}}\right] = (P \otimes \eta)\left[\frac{f}{E^*_{P \otimes \eta, \mathcal{Q} \otimes \eta}}\right]$$

and since $f \in \mathcal{E}_{\mathcal{Q} \otimes H}$ it is in $\mathcal{E}_{\mathcal{Q} \otimes \eta}$ and the numeraire property gives that this ratio is less than or equal to 1. We conclude that $E^*_{P,\mathcal{Q}} E^*_{P \otimes \kappa, P \otimes H}$ satisfies the numeraire property.

We then have

$$(P \otimes \kappa)\left[\log \frac{f}{E^*_{P,\mathcal{Q}} E^*_{P\otimes\kappa, P\otimes H}}\right] \le \log\left((P \otimes \kappa)\left[\frac{f}{E^*_{P,\mathcal{Q}} E^*_{P\otimes\kappa, P\otimes H}}\right]\right) \le 0\,.$$

We conclude that

$$P[\log f] \le P\left[\log E^*_{P,\mathcal{Q}}\right] + (P \otimes \kappa)\left[\log E^*_{P\otimes\kappa, P\otimes H}\right]\,.$$

Taking the supremum over $f \in \mathcal{E}_{\mathcal{Q}\otimes H}$, we obtain the desired inequality. □

Definition 2.44. A jointly measurable function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_{+,\infty}$ is a conditional e-variable for a set of Markov kernels $H \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})$ if $f(x, \cdot)$ is in $\mathcal{E}_{H(x)}$ for all $x \in \mathcal{X}$ (in which $H(x) = \{\eta(x) \mid \eta \in H\}$). We denote the set of conditional e-variables for H by $\mathcal{E}_H$.

Definition 2.45. Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel and $H \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})$ a set of Markov kernels. A conditional e-variable f for H is a conditional numeraire for (κ, H) if for all $x \in \mathcal{X}$, the function $f(x, \cdot)$ is a numeraire for $(\kappa(x), H(x))$. We denote the conditional numeraire for (κ, H) by $E^*_{\kappa,H}$ (if it exists).

As we will see, conditional numeraires are not always guaranteed to exist.

Question: is there always a P-a.e. conditional numeraire? That is, a jointly measurable function f such that $f(x, \cdot)$ is a numeraire for $(\kappa(x), H(x))$ for only P-almost all $x \in \mathcal{X}$ instead of all $x \in \mathcal{X}$?

Lemma 2.46. *If there exists a conditional numeraire for $(\kappa, \{\eta\})$, then $1/E^*_{\kappa,\{\eta\}}$ is a Radon-Nikodym derivative for η with respect to κ.*

Proof. $1/E^*_{\kappa,\{\eta\}}$ is jointly measurable and satisfies for all $x \in \mathcal{X}$, $1/E^*_{\kappa,\{\eta\}}(x, \cdot) =_{a.e.} \frac{d\eta(x)}{d\kappa(x)}$ since that's the numeraire for a singleton. That's the definition of a Radon-Nikodym derivative of kernels. □

The last lemma implies that $E^*_{\kappa,H}$ will not always exist. Indeed, Radon-Nikodym derivatives of kernels do not always exist, although weak hypotheses on $\mathcal{X}$ or $\mathcal{Y}$ can ensure that they do. $\mathcal{X}$ being countable or $\mathcal{Y}$ having countably generated sigma-algebra (for example $\mathcal{Y}$ standard Borel) is enough.

Theorem 2.47. *If a conditional numeraire exists for (κ, H), then $E^*_{P\otimes\kappa, \mathcal{Q}\otimes H} =_{a.e.} E^*_{P,\mathcal{Q}} E^*_{\kappa,H}$.*

Proof. $E^*_{\kappa,H}$ is a numeraire for $(P \otimes \kappa, \{P\} \otimes H)$, so we get the numeraire property as in Theorem 2.43. We now prove that $E^*_{P,\mathcal{Q}} E^*_{\kappa,H} \in \mathcal{E}_{\mathcal{Q}\otimes H}$. Let $Q \in \mathcal{Q}$ and $\eta \in H$.

$$(Q \otimes \eta)\left[E^*_{P,\mathcal{Q}} E^*_{\kappa,H}\right] = Q\left[x \mapsto E^*_{P,\mathcal{Q}}(x)\eta(x)\left[E^*_{\kappa(x),H(x)}\right]\right]\,.$$

For all x, $\eta(x)\left[E^*_{\kappa(x),H(x)}\right] \le 1$, since $E^*_{\kappa(x),H(x)}$ is an e-variable for $H(x)$.

$$(Q \otimes \eta)\left[E^*_{P,\mathcal{Q}} E^*_{\kappa,H}\right] \le Q\left[x \mapsto E^*_{P,\mathcal{Q}}(x)\right] \le 1\,.$$

□

Lemma 2.48. *Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel and $H \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})$ a set of Markov kernels. If $\mathcal{X}$ is countable with measurable singletons then there exists a conditional numeraire for (κ, H).*

Proof. For each $x \in \mathcal{X}$, we can apply Theorem 2.39 to the Markov kernel $\kappa(x)$ and the set of kernels $H(x)$ to obtain a numeraire $E^*_{\kappa(x),H(x)}$. If $\mathcal{X}$ is countable, we can simply take the function $E^*_{\kappa,H}(x,y) = E^*_{\kappa(x),H(x)}(y)$, and the countability ensures the joint measurability. □

Theorem 2.49. *Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel and $H \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})$ a set of Markov kernels. If either $\mathcal{X}$ is countable or $\mathcal{Y}$ is countably generated, then there exists a conditional numeraire for (κ, H).*

Proof. The case of countable $\mathcal{X}$ is covered by Lemma 2.48. (TODO: only for measurable singletons though)

TODO: $\kappa \ll H$ hyp?

Let's now consider the case where $\mathcal{Y}$ is countably generated. Then there exists a sequence of finer and finer partitions of $\mathcal{Y}$ into finitely many measurable sets $(A_{n,m})_{n\in\mathbb{N}, m \le 2^n}$ with $\mathcal{Y} = \bigcup_{m\le 2^n} A_{n,m}$ for all n, and such that the sigma-algebra on $\mathcal{Y}$ is the σ-algebra generated by the sets $A_{n,m}$. Let $\mathcal{F}$ be the filtration generated by the successive partitions, that is, $\mathcal{F}_n = \sigma(\{A_{n,m} \mid m \le 2^n\})$.

TODO □

2.4.3 Reverse information projection and duality.

Definition 2.50. Let $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. The reverse information projection (RIPr) of P onto $\mathcal{Q}$ is the finite measure $P^*_{\mathcal{Q}}$ absolutely continuous with respect to P such that

$$\frac{dP^*_{\mathcal{Q}}}{dP} = \frac{1}{E^*_{P,\mathcal{Q}}} .$$

It satisfies $P^*_{\mathcal{Q}}[\mathcal{X}] \le 1$ and the inequality can be strict.

Definition 2.51. For a set of probability measures $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$, the effective set of measures $\mathcal{Q}_{\text{eff}}$ is the set $\{\mu \in \mathcal{M}_+(\mathcal{X}) \mid \forall f \in \mathcal{E}_{\mathcal{Q}},\ \mu[f] \le 1\}$.

We could also define the effective set of measures for randomized E-variables, but it actually gives the same set.

Lemma 2.52. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be a set of probability measures. The effective set $\mathcal{Q}_{\text{eff}}$ is convex and satisfies $0 \in \mathcal{Q}_{\text{eff}}$ and $\mathcal{Q} \subseteq \mathcal{Q}_{\text{eff}}$.*

Proof. □

Lemma 2.53. *Let $Q \in \mathcal{P}(\mathcal{X})$ be a probability measure. Then $\{Q\}_{\text{eff}} = \{Q' \in \mathcal{M}(\mathcal{X}) \mid Q' \le Q\}$, in which $Q' \le Q$ means that $Q'(A) \le Q(A)$ for all measurable sets $A \subseteq \mathcal{X}$.*

Proof.

$$\begin{aligned}
\{Q\}_{\text{eff}} &= \{Q' \in \mathcal{M}(\mathcal{X}) \mid \forall f,\ (\forall Q'' \in \{Q\}, Q''[f] \le 1) \implies Q'[f] \le 1\} \\
&= \{Q' \in \mathcal{M}(\mathcal{X}) \mid \forall f,\ Q[f] \le 1 \implies Q'[f] \le 1\}
\end{aligned}$$

This is the set of finite measures $\{Q' \mid Q' \le Q\}$. Indeed, if $Q' \le Q$, then for all f with $Q[f] \le 1$, $Q'[f] \le Q[f] \le 1$. Conversely, if Q' is such that $(\forall f,\ Q[f] \le 1 \implies Q'[f] \le 1)$, then for any measurable set A, we can take the measurable function $f = Q(A)^{-1} \cdot \mathbb{I}_A$ (in which $0^{-1} = \infty$) to obtain $Q'(A) \le Q(A)$. □

Lemma 2.54. *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ and let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be a Markov kernel. Then $\kappa \circ \mathcal{Q}_{\text{eff}} \subseteq (\kappa \circ \mathcal{Q})_{\text{eff}}$.*

Proof.

$$\kappa \circ \mathcal{Q}_{\text{eff}} = \{\kappa \circ Q \mid \forall f, (\forall Q' \in \mathcal{Q}, Q'[f] \le 1) \implies Q[f] \le 1\}$$
$$(\kappa \circ \mathcal{Q})_{\text{eff}} = \{R \mid \forall f, (\forall Q' \in \mathcal{Q}, (\kappa \circ Q')[f] \le 1) \implies R[f] \le 1\}$$

Let $\kappa \circ Q \in \kappa \circ \mathcal{Q}_{\text{eff}}$ and f such that $\forall Q' \in \mathcal{Q}, (\kappa \circ Q')[f] \le 1$. That is, $\forall Q' \in \mathcal{Q}, Q'[x \mapsto \kappa(x)[f]] \le 1$. Then since $Q \in \mathcal{Q}_{\text{eff}}$, we have $Q[x \mapsto \kappa(x)[f]] \le 1$. We have proved that $\kappa \circ Q \in (\kappa \circ \mathcal{Q})_{\text{eff}}$. □

Lemma 2.55. *Let $\mathcal{Q}_1 \subseteq \mathcal{Q}_2 \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Then $\mathcal{Q}_{1,\text{eff}} \subseteq \mathcal{Q}_{2,\text{eff}}$.*

Proof. Since $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ we have $\mathcal{E}_{\mathcal{Q}_2} \subseteq \mathcal{E}_{\mathcal{Q}_1}$ (Lemma 2.2). Then

$$\begin{aligned}
\mathcal{Q}_{1,\text{eff}} &= \{\mu \in \mathcal{M}_+(\mathcal{X}) \mid \forall f \in \mathcal{E}_{\mathcal{Q}_1},\ \mu[f] \le 1\} \\
&\subseteq \{\mu \in \mathcal{M}_+(\mathcal{X}) \mid \forall f \in \mathcal{E}_{\mathcal{Q}_2},\ \mu[f] \le 1\} \\
&= \mathcal{Q}_{2,\text{eff}}\ .
\end{aligned}$$

□

Let $\mathrm{KL}(P, Q)$ be the Kullback-Leibler divergence from P to Q, defined for finite measures as

$$\mathrm{KL}(P, Q) = \begin{cases} P[\log \frac{dP}{dQ}] & \text{if } P \ll Q \\ +\infty & \text{otherwise} \end{cases} .$$

Note that we don't correct the divergence if P and Q are not probability measures.

Theorem 2.56. *Let $P \in \mathcal{P}(\mathcal{X})$ and $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$. If $P \ll \mathcal{Q}$ then*

$$P[\log E^*_{P,\mathcal{Q}}] = \sup_{f \in \mathcal{E}_{\mathcal{Q}}} P[\log f] = \inf_{\mu \in \mathcal{Q}_{\text{eff}}} \mathrm{KL}(P, \mu) = \mathrm{KL}(P, P^*_{\mathcal{Q}})\ .$$

Proof.

□

Lemma 2.57. *Let $P, Q \in \mathcal{P}(\mathcal{X})$ be two probability measures with $P \ll Q$. Then*

$$\sup_{f \in \mathcal{E}_{\{Q\}}} P[\log f] = \mathrm{KL}(P, Q)\ .$$

Proof. Since KL is non-increasing in the second argument,

$$\inf_{Q' \le Q} \mathrm{KL}(P, Q') = \mathrm{KL}(P, Q)\ .$$

Since $\{Q\}_{\text{eff}} = \{Q' \in \mathcal{M}(\mathcal{X}) \mid Q' \le Q\}$ (Lemma 2.53), we obtained $\mathrm{KL}(P, Q) = \inf_{Q' \in \{Q\}_{\text{eff}}} \mathrm{KL}(P, Q')$. By the duality of numeraire and KL divergence (Theorem 2.56), we have

$$\sup_{f \in \mathcal{E}_{\{Q\}}} P[\log f] = \inf_{Q' \in \{Q\}_{\text{eff}}} \mathrm{KL}(P, Q') = \mathrm{KL}(P, Q)\ .$$

□

KL data processing from e-variables The data processing inequality for the numeraire utility implies the data processing inequality for KL divergences.

Theorem 2.58. *Let $P, Q \in \mathcal{P}(\mathcal{X})$. Then for all Markov kernels $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$,*

$$\mathrm{KL}(P, Q) \geq \mathrm{KL}(\kappa \circ P, \kappa \circ Q) \, .$$

Proof. By Lemma 2.57, $\mathrm{KL}(P, Q) = \sup_{f \in \mathcal{E}_{\{Q\}}} P[\log f]$. Then by Lemma 2.24, we have

$$\sup_{f \in \mathcal{E}_{\{Q\}}} P[\log f] \geq \sup_{g \in \mathcal{E}_{\{\kappa \circ Q\}}} (\kappa \circ P)[\log g] \, .$$

And that last term is $\mathrm{KL}(\kappa \circ P, \kappa \circ Q)$, again by Lemma 2.57. $\square$

Chapter 3

e-Rényi and e-Chernoff divergences

Definition 3.1. The e-Rényi divergence of order $\alpha \in (0,1)$ between two sets of probability measures $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ is defined as

$$\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) = \frac{1}{1-\alpha} \inf_{R \in \mathcal{P}(\mathcal{X})} \left(\alpha \mathcal{U}(R, \mathcal{P}) + (1-\alpha)\mathcal{U}(R, \mathcal{Q})\right) \,,$$

for $\mathcal{U}_{\log}$ the max-utility corresponding to the logarithmic utility.

Definition 3.2.

$$\mathrm{eC}_\alpha(\mathcal{P}, \mathcal{Q}) = \inf_{R \in \mathcal{P}(\mathcal{X})} \max\left\{\mathcal{U}(R, \mathcal{P}), \mathcal{U}_{\log}(R, \mathcal{Q})\right\} \,.$$

Lemma 3.3. *Let $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$. Then, for all $\alpha \in (0,1)$,*

$$\mathrm{eR}_\alpha(\mathcal{P}', \mathcal{Q}') \geq \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) \,.$$

Proof. □

Lemma 3.4. *Let $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$. Then*

$$\mathrm{eC}_\alpha(\mathcal{P}', \mathcal{Q}') \geq \mathrm{eC}_\alpha(\mathcal{P}, \mathcal{Q}) \,.$$

Proof. □

Lemma 3.5. *Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be Markov kernel. Then*

$$\mathrm{eR}_\alpha(\kappa \circ \mathcal{P}, \kappa \circ \mathcal{Q}) \leq \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) \,.$$

Proof. □

Lemma 3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a measurable map. Then*

$$\mathrm{eR}_\alpha(f_* \mathcal{P}, f_* \mathcal{Q}) \leq \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) \,.$$

Proof. □

Lemma 3.7. *Let $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ be Markov kernel. Then*

$$\mathrm{eC}_\alpha(\kappa \circ \mathcal{P}, \kappa \circ \mathcal{Q}) \le \mathrm{eC}_\alpha(\mathcal{P}, \mathcal{Q}) .$$

Proof. □

Lemma 3.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a measurable map. Then*

$$\mathrm{eC}_\alpha(f_* \mathcal{P}, f_* \mathcal{Q}) \le \mathrm{eC}_\alpha(\mathcal{P}, \mathcal{Q}) .$$

Proof. □

Lemma 3.9. *For $\alpha \in (0, 1)$, if $\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) < \infty$, then*

$$\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) = \frac{1}{1-\alpha} \inf_{R \in \mathcal{P}(\mathcal{X}),\, R \ll \mathcal{P},\, R \ll \mathcal{Q}} \left(\alpha \mathcal{U}_{\log}(R, \mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(R, \mathcal{Q})\right) .$$

That is, the infimum defining $\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q})$ can be restricted to measures R that are absolutely continuous with respect to both $\mathcal{P}$ and $\mathcal{Q}$.

TODO: can we drop the finiteness assumption?

Proof. If R is not absolutely continuous with respect to $\mathcal{P}$, then $\mathcal{U}_{\log}(R, \mathcal{P}) = \infty$ by Lemma 2.19. Similarly, if R is not absolutely continuous with respect to $\mathcal{Q}$, then $\mathcal{U}_{\log}(R, \mathcal{Q}) = \infty$. Hence, in both cases, the term $\alpha \mathcal{U}_{\log}(\mathcal{R}, \mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(\mathcal{R}, \mathcal{Q})$ is infinite and does not contribute to the infimum defining $\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q})$. □

Corollary 3.10. *For $\alpha \in (0, 1)$, if $\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) < \infty$ and there exists a measurable set A with $P(A^c) = 0$ and $Q(A^c) = 0$ for all $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, then*

$$\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) = \frac{1}{1-\alpha} \inf_{R \in \mathcal{P}(\mathcal{X}),\, R(A^c)=0} \left(\alpha \mathcal{U}_{\log}(R, \mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(R, \mathcal{Q})\right) .$$

That is, the infimum defining $\mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q})$ can be restricted to measures R supported on A.

Proof. We apply Lemma 3.9 and note that any measure R that is absolutely continuous with respect to both $\mathcal{P}$ and $\mathcal{Q}$ must satisfy $R(A^c) = 0$. □

Lemma 3.11. *Let $\phi : \mathcal{X} \to \mathcal{Y}$ be a measurable embedding. Then*

$$\mathrm{eR}_\alpha(\phi_* \mathcal{P}, \phi_* \mathcal{Q}) = \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) .$$

Proof. By Lemma 3.6, we have $\mathrm{eR}_\alpha(\phi_* \mathcal{P}, \phi_* \mathcal{Q}) \le \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q})$. Then either both sides are infinite and we are done, or the left hand side is finite.

Let $A = \phi(\mathcal{X}) \subseteq \mathcal{Y}$. Then for all $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, we have $\phi_* P(A^c) = 0$ and $\phi_* Q(A^c) = 0$. By Corollary 3.10, we can restrict the infimum defining $\mathrm{eR}_\alpha(\phi_* \mathcal{P}, \phi_* \mathcal{Q})$ to measures supported on A. Since ϕ is an embedding, every measure supported on A is of the form $\phi_* R$ for some $R \in \mathcal{P}(\mathcal{X})$ (take $R = \phi_*^{-1} R'$, in which ϕ^{-1} is a partial inverse on the range of ϕ).

$$\begin{aligned}\mathrm{eR}_\alpha(\phi_*\mathcal{P}, \phi_*\mathcal{Q}) &= \frac{1}{1-\alpha} \inf_{R' \in \mathcal{P}(\mathcal{Y}),\, R'(A^c)=0} \left(\alpha \mathcal{U}_{\log}(R', \phi_*\mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(R', \phi_*\mathcal{Q})\right) \\ &= \frac{1}{1-\alpha} \inf_{R \in \mathcal{P}(\mathcal{X})} \left(\alpha \mathcal{U}_{\log}(\phi_* R, \phi_*\mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(\phi_* R, \phi_*\mathcal{Q})\right) .\end{aligned}$$

Using then Lemma 2.26, we get

$$\begin{aligned}\mathrm{eR}_\alpha(\phi_*\mathcal{P}, \phi_*\mathcal{Q}) &= \frac{1}{1-\alpha} \inf_{R \in \mathcal{P}(\mathcal{X})} \left(\alpha \mathcal{U}_{\log}(R, \mathcal{P}) + (1-\alpha)\mathcal{U}_{\log}(R, \mathcal{Q})\right) \\ &= \mathrm{eR}_\alpha(\mathcal{P}, \mathcal{Q}) .\end{aligned}$$

□

Lemma 3.12.

$$\mathcal{U}(P_1 \otimes P_2, \mathcal{Q}_1 \otimes \mathcal{Q}_2) = \mathcal{U}(P_1, \mathcal{Q}_1) + \mathcal{U}(P_2, \mathcal{Q}_2) .$$

Proof. □

Lemma 3.13.

$$\mathrm{eR}_\alpha(\mathcal{P}_1 \otimes \mathcal{P}_2, \mathcal{Q}_1 \otimes \mathcal{Q}_2) = \mathrm{eR}_\alpha(\mathcal{P}_1, \mathcal{Q}_1) + \mathrm{eR}_\alpha(\mathcal{P}_2, \mathcal{Q}_2) .$$

Proof.

□

Lemma 3.14.

$$\mathrm{eC}_\alpha(\mathcal{P}_1 \otimes \mathcal{P}_2, \mathcal{Q}_1 \otimes \mathcal{Q}_2) \le \mathrm{eC}_\alpha(\mathcal{P}_1, \mathcal{Q}_1) + \mathrm{eC}_\alpha(\mathcal{P}_2, \mathcal{Q}_2) .$$

Proof.

□

Lemma 3.15. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Suppose that there exists a measurable map $\phi : \mathcal{X} \to \mathcal{X}$ such that $\phi \circ \phi = \mathrm{id}$ and $\phi_*\mathcal{P} = \mathcal{Q}$. Then*

$$\mathrm{eR}_{1/2}(\mathcal{P}, \mathcal{Q}) = 2 \inf_{R \in \mathcal{P}(\mathcal{X}),\, \phi_* R = R} \mathcal{U}_{\log}(R, \mathcal{P}) .$$

Proof. First consider an arbitrary $R \in \mathcal{P}(\mathcal{X})$. Then $R' = \frac{1}{2}(R + \phi_* R)$ satisfies $\phi_* R' = R'$ and by convexity (Lemma 2.20) and Lemma 2.27,

$$\begin{aligned}\frac{1}{2}\mathcal{U}_{\log}(R', \mathcal{P}) + \frac{1}{2}\mathcal{U}_{\log}(R', \mathcal{Q}) &\le \frac{1}{2}\left(\frac{1}{2}\mathcal{U}_{\log}(R, \mathcal{P}) + \frac{1}{2}\mathcal{U}_{\log}(R, \mathcal{Q})\right) \\ &\quad + \frac{1}{2}\left(\frac{1}{2}\mathcal{U}_{\log}(\phi_* R, \mathcal{P}) + \frac{1}{2}\mathcal{U}_{\log}(\phi_* R, \mathcal{Q})\right) \\ &= \frac{1}{2}\mathcal{U}_{\log}(R, \mathcal{P}) + \frac{1}{2}\mathcal{U}_{\log}(R, \mathcal{Q}) .\end{aligned}$$

Hence we can restrict the infimum defining $\mathrm{eR}_{1/2}(\mathcal{P}, \mathcal{Q})$ to measures R such that $\phi_* R = R$. For such measures, by Lemma 2.27, $\mathcal{U}_{\log}(R, \mathcal{Q}) = \mathcal{U}_{\log}(\phi_* R, \mathcal{P}) = \mathcal{U}_{\log}(R, \mathcal{P})$ and we get the result. □

Lemma 3.16. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Suppose that there exists a measurable map $\phi : \mathcal{X} \to \mathcal{X}$ such that $\phi \circ \phi = \mathrm{id}$ and $\phi_* \mathcal{P} = \mathcal{Q}$. Then*

$$\mathrm{eC}(\mathcal{P}, \mathcal{Q}) = \inf_{R \in \mathcal{P}(\mathcal{X}),\, \phi_* R = R} \mathcal{U}_{\log}(R, \mathcal{P}) \,.$$

Proof. Similar to the proof of Lemma 3.15. □

Lemma 3.17. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Suppose that there exists a measurable map $\phi : \mathcal{X} \to \mathcal{X}$ such that $\phi \circ \phi = \mathrm{id}$ and $\phi_* \mathcal{P} = \mathcal{Q}$. Then*

$$\mathrm{eR}_{1/2}(\mathcal{P}, \mathcal{Q}) = 2\, \mathrm{eC}(\mathcal{P}, \mathcal{Q}) \,.$$

Proof. Use Lemmas 3.15 and 3.16. □

Chapter 4

Lower bounds on e-Rényi and e-Chernoff divergences

4.1 Separation lower bound

Lemma 4.1. *Let* $a \le b \in [0,1]$. *Let* $\mathcal{D}_{\le a} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \le a\}$ *and* $\mathcal{D}_{\ge b} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \ge b\}$ *be sets of distributions on* $[0,1]$ *with mean constraints. Let* $B_{\le a} = \{\mathrm{Ber}(p) \mid p \le a\}$ *and* $B_{\ge a} = \{\mathrm{Ber}(p) \mid p \ge a\}$ *be sets of Bernoulli distributions with the same constraints. Then*

$$\mathrm{eR}_{1/2}(\mathcal{D}_{\le a}, \mathcal{D}_{\ge b}) = \mathrm{eR}_{1/2}(B_{\le a}, B_{\ge b})\ .$$

Proof. First, by Lemma 3.3, since $B_{\le a} \subseteq \mathcal{D}_{\le a}$ and $B_{\ge b} \subseteq \mathcal{D}_{\ge b}$, we have

$$\mathrm{eR}_{1/2}(\mathcal{D}_{\le a}, \mathcal{D}_{\ge b}) \le \mathrm{eR}_{1/2}(B_{\le a}, B_{\ge b})\ .$$

For the reverse inequality, we use the data processing inequality for e-Rényi divergence (Lemma 3.5). Let $\kappa : [0,1] \rightsquigarrow \{0,1\}$ be the Markov kernel defined by $\kappa(x) = \mathrm{Ber}(x)$ (the Bernoulli distribution with parameter x). Then for $P \in \mathcal{P}([0,1])$, $\kappa \circ P$ is a Bernoulli distribution with parameter $P[\mathrm{id}]$, and thus $\kappa \circ \mathcal{D}_{\le a} = B_{\le a}$ and $\kappa \circ \mathcal{D}_{\ge b} = B_{\ge b}$. By Lemma 3.5,

$$\mathrm{eR}_{1/2}(B_{\le a}, B_{\ge b}) \le \mathrm{eR}_{1/2}(\mathcal{D}_{\le a}, \mathcal{D}_{\ge b})\ .$$

□

Lemma 4.2. *Let* $\delta \in [0,1/2]$ *and let* $B_{\le \delta} = \{\mathrm{Ber}(p) \mid p \le \delta\}$. *Then*

$$\mathcal{E}_{B_{\le\delta}} = \left\{ f : \{0,1\} \to \mathbb{R}_{+,\infty} \text{ measurable} \mid \exists \lambda \in \left[0, \frac{1}{\delta}\right],\ \forall x \in \{0,1\},\ f(x) \le 1 + \lambda(x - \delta) \right\}\ .$$

Proof. It is easy to check that the set on the right is included in $\mathcal{E}_{B_{\le\delta}}$.

The constraint on an e-variable f for $B_{\le\delta}$ is that for all $p \le \delta$,

$$(1-p)f(0) + pf(1) \le 1\ .$$

For $p = 0$ and $p = \delta$ we thus get the two constraints

$$f(0) \le 1\ ,$$
$$f(1) \le 1 + \frac{1 - f(0)}{\delta}(1 - \delta)\ .$$

Let $\lambda = \frac{1-f(0)}{\delta}$. Then $\lambda \in [0, \frac{1}{\delta}]$ and the second constraint can be rewritten as $f(1) \le 1+\lambda(1-\delta)$. We need to check that we also have $f(0) \le 1 + \lambda(0-\delta)$, but the expression on the right is equal to $f(0)$ by definition of λ. $\square$

Lemma 4.3. *Let $\delta \in [0, 1/2]$ and let $B_{\le\delta} = \{\mathrm{Ber}(p) \mid p \le \delta\}$. Then*

$$\mathcal{U}_{\log}(\mathrm{Ber}(1/2), B_{\le\delta}) = \frac{1}{2}\log\frac{1}{4\delta(1-\delta)} .$$

Proof. We first compute $\mathcal{E}_{B_{\le\delta}}$ with Lemma 4.2. We thus know that we can restrict the supremum over $f \in \mathcal{E}_{B_{\le\delta}}$ to functions of the form $f(x) = 1 + \lambda(x-\delta)$ for some $\lambda \in [0, \frac{1}{\delta}]$.

$$\begin{aligned}\mathcal{U}_{\log}(\mathrm{Ber}(1/2), B_{\le\delta}) &= \sup_{\lambda\in[0,\frac{1}{\delta}]} \frac{1}{2}\log(1+\lambda(0-\delta)) + \frac{1}{2}\log(1+\lambda(1-\delta)) \\ &= \frac{1}{2}\log \sup_{\lambda\in[0,\frac{1}{\delta}]} \left((1-\lambda\delta)(1+\lambda(1-\delta))\right) .\end{aligned}$$

The function $(1-\lambda\delta)(1+\lambda(1-\delta))$ reaches its maximum at $\lambda^* = \frac{1-2\delta}{2\delta(1-\delta)}$ which is in the interval $[0, \frac{1}{\delta}]$ since $\delta \in [0, 1/2]$. Its value at the optimum is $\frac{1}{4\delta(1-\delta)}$. $\square$

Lemma 4.4. *Let $\delta \in [0, 1/2]$. Let $B_{\le\delta} = \{\mathrm{Ber}(p) \mid p \le \delta\}$ and $B_{\ge 1-\delta} = \{\mathrm{Ber}(p) \mid p \ge 1-\delta\}$ be sets of Bernoulli distributions with mean constraints. Then*

$$\mathrm{eR}_{1/2}(B_{\le\delta}, B_{\ge1-\delta}) = \log\frac{1}{4\delta(1-\delta)} .$$

Proof. By Lemma 3.15 for the map $\phi : \{0,1\} \to \{0,1\}$ defined by $\phi(0) = 1$ and $\phi(1) = 0$, we have

$$\begin{aligned}\mathrm{eR}_{1/2}(B_{\le\delta}, B_{\ge1-\delta}) &= 2 \inf_{R\in\mathcal{P}(\{0,1\}),\, \phi_*R=R} \mathcal{U}_{\log}(R, B_{\le\delta}) \\ &= 2\mathcal{U}_{\log}\left(\mathrm{Ber}(1/2), B_{\le\delta}\right) .\end{aligned}$$

We now use Lemma 4.3 to conclude. $\square$

Lemma 4.5. *Let $\delta \in [0, 1/2]$. Let $B_{\le\delta} = \{\mathrm{Ber}(p) \mid p \le \delta\}$ and $B_{\ge 1-\delta} = \{\mathrm{Ber}(p) \mid p \ge 1-\delta\}$ be sets of Bernoulli distributions with mean constraints. Then*

$$\mathrm{eC}(B_{\le\delta}, B_{\ge1-\delta}) = \frac{1}{2}\log\frac{1}{4\delta(1-\delta)} .$$

Proof. By Lemma 3.17, we have

$$\mathrm{eC}(B_{\le\delta}, B_{\ge1-\delta}) = \frac{1}{2}\mathrm{eR}_{1/2}(B_{\le\delta}, B_{\ge1-\delta}) .$$

We now use Lemma 4.4 to conclude. $\square$

Lemma 4.6. *Let $\delta \in [0, 1/2]$. Let $\mathcal{D}_{\le\delta} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \le \delta\}$ and $\mathcal{D}_{\ge1-\delta} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \ge 1-\delta\}$ be sets of distributions on $[0,1]$ with mean constraints. Then*

$$\mathrm{eR}_{1/2}(\mathcal{D}_{\le\delta}, \mathcal{D}_{\ge1-\delta}) = \log\frac{1}{4\delta(1-\delta)} .$$

Proof. By Lemma 4.1, we have

$$\mathrm{eR}_{1/2}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta}) = \mathrm{eR}_{1/2}(B_{\leq\delta}, B_{\geq 1-\delta})\,.$$

We now use Lemma 4.4 to conclude. □

Lemma 4.7. *Let $\delta \in [0, 1/2]$. Let $\mathcal{D}_{\leq\delta} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \leq \delta\}$ and $\mathcal{D}_{\geq 1-\delta} = \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \geq 1-\delta\}$ be sets of distributions on $[0,1]$ with mean constraints. Then*

$$\mathrm{eC}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta}) = \frac{1}{2}\log\frac{1}{4\delta(1-\delta)}\,.$$

Proof. By Lemma 3.17, we have

$$\mathrm{eC}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta}) = \frac{1}{2}\,\mathrm{eR}_{1/2}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta})\,.$$

We now use Lemma 4.6 to conclude. □

Theorem 4.8. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Suppose that there exists a measurable function $f_0 : \mathcal{X} \to \mathbb{R}_{+,\infty}$ which is less than 1 and $\delta \in [0, 1/2]$ such that*

$$\forall P \in \mathcal{P},\ P[f_0] \leq \delta\,,$$
$$\forall Q \in \mathcal{Q},\ Q[f_0] \geq 1-\delta\,.$$

Then

$$\mathrm{eR}_{1/2}(\mathcal{P}, \mathcal{Q}) \geq \log\frac{1}{4\delta(1-\delta)}\,.$$

Proof. By Lemma 3.6 for the function f_0, we have

$$\mathrm{eR}_{1/2}(\mathcal{P}, \mathcal{Q}) \geq \mathrm{eR}_{1/2}(f_0 \circ \mathcal{P}, f_0 \circ \mathcal{Q})\,.$$

$f_0 \circ \mathcal{P} \subseteq \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \leq \delta\}$ and $f_0 \circ \mathcal{Q} \subseteq \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \geq 1-\delta\}$. By Lemma 3.3, we thus have

$$\mathrm{eR}_{1/2}(f_0 \circ \mathcal{P}, f_0 \circ \mathcal{Q}) \geq \mathrm{eR}_{1/2}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta})\,.$$

We now use Lemma 4.6 to conclude. □

Theorem 4.9. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathcal{P}(\mathcal{X})$ be two sets of probability measures. Suppose that there exists a measurable function $f_0 : \mathcal{X} \to \mathbb{R}_{+,\infty}$ which is less than 1 and $\delta \in [0, 1/2]$ such that*

$$\forall P \in \mathcal{P},\ P[f_0] \leq \delta\,,$$
$$\forall Q \in \mathcal{Q},\ Q[f_0] \geq 1-\delta\,.$$

Then

$$\mathrm{eC}(\mathcal{P}, \mathcal{Q}) \geq \frac{1}{2}\log\frac{1}{4\delta(1-\delta)}\,.$$

Proof. By Lemma 3.8 for the function f_0, we have

$$\mathrm{eC}(\mathcal{P}, \mathcal{Q}) \geq \mathrm{eC}(f_0 \circ \mathcal{P}, f_0 \circ \mathcal{Q})\,.$$

$f_0 \circ \mathcal{P} \subseteq \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \leq \delta\}$ and $f_0 \circ \mathcal{Q} \subseteq \{P \in \mathcal{P}([0,1]) \mid P[\mathrm{id}] \geq 1-\delta\}$. By Lemma 3.4, we thus have

$$\mathrm{eC}(f_0 \circ \mathcal{P}, f_0 \circ \mathcal{Q}) \geq \mathrm{eC}(\mathcal{D}_{\leq\delta}, \mathcal{D}_{\geq 1-\delta})\,.$$

We now use Lemma 4.7 to conclude. □

4.2 Data processing inequality through KL duality

$$\begin{aligned}
\sup_{f \in \mathcal{E}_{\mathcal{Q}}} P[\log f] &= \inf_{Q \in \mathcal{Q}_{\mathrm{eff}}} \mathrm{KL}(P, Q) \\
&\geq \inf_{Q \in \mathcal{Q}_{\mathrm{eff}}} \mathrm{KL}(\kappa \circ P, \kappa \circ Q) \\
&= \inf_{Q' \in \kappa \circ \mathcal{Q}_{\mathrm{eff}}} \mathrm{KL}(\kappa \circ P, Q') \\
&\geq \inf_{Q' \in (\kappa \circ \mathcal{Q})_{\mathrm{eff}}} \mathrm{KL}(\kappa \circ P, Q') \\
&= \sup_{g \in \mathcal{E}_{\kappa \circ \mathcal{Q}}} (\kappa \circ P)[\log g] \; .
\end{aligned}$$

The first inequality is the data processing inequality for KL divergence. The second inequality comes from $\kappa \circ \mathcal{Q}_{\mathrm{eff}} \subseteq (\kappa \circ \mathcal{Q})_{\mathrm{eff}}$, which we proved in Lemma 2.54.

Bibliography

[LRR24] Martin Larsson, Aaditya Ramdas, and Johannes Ruf, *The numeraire e-variable and reverse information projection*, arXiv preprint arXiv:2402.18810 (2024).