

Sequential Learning

Lecture 5 : Beyond Value-Based Methods

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Reminder

Until now we have seen **Value-Based methods** , that learn

$$Q(s, a)$$

an estimate of the optimal Q-Value function

$$\begin{aligned} Q^*(s, a) &= \max_{\pi} Q^{\pi}(s, a) \\ &= \max_{\pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \mid s_1 = s, a_1 = a \right] \end{aligned}$$

→ our guess for the optimal policy is then $\pi = \text{greedy}(Q)$:

$$\pi(s) = \operatorname{argmax}_{a \in \mathcal{A}} Q(s, a)$$

(a deterministic policy)

Outline

- 1** Optimizing Over Policies
- 2 Policy Gradients
- 3 The REINFORCE algorithm
- 4 Advantage Actor Critic

Optimizing over policies ?

We could try to

$$\operatorname{argmax}_{\pi \in \Pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \mid s_1 \sim \rho \right]$$

where

$$\Pi = \{ \text{stationary, deterministic policies } \pi : \mathcal{S} \rightarrow \mathcal{A} \}$$

and ρ is a distribution over first states.

→ intractable !

Idea : relax this optimization problem by searching over a (smoothly) **parameterized** set of **stochastic** policies.

A new objective

- ▶ parametric family of **stochastic** policies $\{\pi_\theta\}_{\theta \in \Theta}$
- ▶ $\pi_\theta(a|s)$: probability of choosing a in s , given θ
- ▶ $\theta \mapsto \pi_\theta(a|s)$ is assumed to be **differentiable**

Goal : find θ that maximizes

$$J(\theta) = \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \mid s_1 \sim \rho \right]$$

over the parameter space Θ .

Idea : use **gradient ascent**

- How to compute the gradient $\nabla_\theta J(\theta)$?
- How to estimate it using trajectories?

Warm-up : Computing gradients

- ▶ $f : \mathcal{X} \rightarrow \mathbb{R}$ is a (non differentiable) function
- ▶ $\{p_\theta\}_{\theta \in \Theta}$ is a set of probability distributions over \mathcal{X}

$$J(\theta) = \mathbb{E}_{X \sim p_\theta} [f(X)]$$

Proposition

$$\nabla_\theta J(\theta) = \mathbb{E}_{X \sim p_\theta} [f(X) \nabla \log p_\theta(X)]$$

Exercise : Prove it !

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Finite-Horizon objective

$$J(\theta) = \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^T \gamma^{t-1} r(s_t, a_t) \mid s_1 \sim \rho \right]$$

for some $\gamma \in (0, 1]$.

- ▶ $\tau = (s_1, a_1, s_2, a_2, \dots, s_T, a_T)$ trajectory of length T
- ▶ π_θ induces a distribution p_θ over trajectories :

$$p_\theta(\tau) = \rho(s_1) \prod_{t=1}^T \pi_\theta(a_t | s_t) p(s_{t+1} | s_t, a_t)$$

- ▶ cumulative discounted reward over the trajectory :

$$R(\tau) := \sum_{t=1}^T \gamma^{t-1} r(s_t, a_t)$$

Finite-Horizon objective

$$J(\theta) = \mathbb{E}_{\tau \sim p_\theta} [R(\tau)]$$

for some $\gamma \in (0, 1]$.

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- ▶ cumulative discounted reward over the trajectory :

$$R(\tau) := \sum_{t=1}^T \gamma^{t-1} r(s_t, a_t)$$

Computing the gradient

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} [R(\tau) \nabla_{\theta} \log p_{\theta}(\tau)]$$

and

$$\begin{aligned} \nabla_{\theta} \log p_{\theta}(\tau) &= \nabla_{\theta} \log \left(\rho(s_1) \prod_{t=1}^T \pi_{\theta}(a_t | s_t) p(s_{t+1} | s_t, a_t) \right) \\ &= \nabla_{\theta} \left(\log \rho(s_1) + \sum_{t=1}^T (\log p(s_{t+1} | s_t, a_t) + \log \pi_{\theta}(a_t | s_t)) \right) \\ &= \sum_{t=1}^T \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \end{aligned}$$

Hence,

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} \left[\sum_{t=1}^T R(\tau) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right]$$

The baseline trick

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} \left[\sum_{t=1}^T R(\tau) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right]$$

In each step t , we may subtract a **baseline function** $b_t(s_1, a_1, \dots, s_t)$, which depends on the beginning of the trajectory (up to s_t), i.e.

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} \left[\sum_{t=1}^T (R(\tau) - b_t(s_1, a_1, \dots, s_t)) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right]$$

Why?

$$\begin{aligned} & \mathbb{E}_{\tau \sim p_{\theta}} [b_t(s_1, a_1, \dots, s_t) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) | s_1, a_1, \dots, s_t] \\ &= b_t(s_1, a_1, \dots, s_t) \sum_{a \in \mathcal{A}} \pi_{\theta}(a | s_t) \nabla_{\theta} \log \pi_{\theta}(a | s_t) \\ &= b_t(s_1, a_1, \dots, s_t) \sum_{a \in \mathcal{A}} \nabla_{\theta} \pi_{\theta}(a | s_t) \\ &= b_t(s_1, a_1, \dots, s_t) \underbrace{\nabla_{\theta} \left(\sum_{a \in \mathcal{A}} \pi_{\theta}(a | s_t) \right)}_{=1} = 0 \end{aligned}$$

Choosing a baseline

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} \left[\sum_{t=1}^T (R(\tau) - b_t(s_1, a_1, \dots, s_t)) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right]$$

A common choice is

$$b_t(s_1, a_1, \dots, s_t) = \sum_{i=1}^{t-1} \gamma^{t-1} r(s_i, a_i)$$

which leads to

$$\begin{aligned} R(\tau) - b_t(s_1, a_1, \dots, s_t) &= \sum_{i=t}^T \gamma^{i-1} r(s_i, a_i) \\ &= \gamma^{t-1} \underbrace{\sum_{i=t}^T \gamma^{i-t} r(s_i, a_i)}_{\text{discounted sum of rewards starting from } s_t} \end{aligned}$$

Policy Gradient Theorem

Using this baseline, we obtain

$$\begin{aligned}\nabla_{\theta} J(\theta) &= \mathbb{E}_{\tau \sim p_{\theta}} \left[\sum_{t=1}^T \gamma^{t-1} \left(\sum_{i=t}^T \gamma^{i-t} r(s_i, a_i) \right) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right] \\ &= \mathbb{E}^{\pi_{\theta}} \left[\sum_{t=1}^T \gamma^{t-1} Q_t^{\pi_{\theta}}(s_t, a_t) \nabla_{\theta} \log \pi_{\theta}(a_t | s_t) \right]\end{aligned}$$

where

$$Q_t^{\pi}(s, a) = \mathbb{E}^{\pi} \left[\sum_{i=t}^T \gamma^{i-t} r(s_i, a_i) \mid s_t = s, a_t = a \right]$$

Policy Gradient Theorem : Infinite Horizon

$$J(\theta) = \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, a_t) \mid s_1 \sim \rho \right]$$

(taking the limit when $T \rightarrow \infty$ of the previous objective)

Policy Gradient Theorem [Sutton et al., 1999]

$$\nabla_\theta J(\theta) = \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} Q^{\pi_\theta}(s_t, a_t) \nabla_\theta \log \pi_\theta(s_t | a_t) \right]$$

where $Q^\pi(s, a)$ is the usual Q-value function of policy π .

Remark : sometimes written

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{(s,a) \sim d^\pi} [Q^{\pi_\theta}(s, a) \nabla_\theta \log \pi_\theta(s|a)]$$

with $d^\pi(s, a) = (1-\gamma) \sum_{t=1}^{\infty} \gamma^{t-1} \mathbb{P}_\pi(S_t = s, A_t = a)$.

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Recap : Exact gradients

► Finite horizon

$$\nabla_{\theta} J(\theta) = \mathbb{E}^{\pi_{\theta}} \left[\sum_{t=1}^T \gamma^{t-1} Q_t^{\pi_{\theta}}(s_t, a_t) \nabla_{\theta} \log \pi_{\theta}(s_t | a_t) \right]$$

► Infinite horizon

$$\nabla_{\theta} J(\theta) = \mathbb{E}^{\pi_{\theta}} \left[\sum_{t=1}^{\infty} \gamma^{t-1} Q^{\pi_{\theta}}(s_t, a_t) \nabla_{\theta} \log \pi_{\theta}(s_t | a_t) \right]$$

- simple formulations to propose **unbiased estimates of the gradients** based on trajectories (almost unbiased for infinite horizon)

REINFORCE

- ▶ Initialize θ arbitrarily
- ▶ In each step, generate N trajectories of length T under π_θ

$$(s_1^{(i)}, a_1^{(i)}, r_1^{(i)}, \dots, s_T^{(i)}, a_T^{(i)}, r_T^{(i)})_{i=1, \dots, N}$$

compute a **Monte-Carlo estimate** of the gradient

$$\widehat{\nabla_\theta J(\theta)} = \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \gamma^t G_t^{(i)} \nabla_\theta \log \pi_\theta(a_t^{(i)} | s_t^{(i)})$$

with $G_t^{(i)} = \sum_{s=t}^T \gamma^{s-t} r_s^{(i)}$.

- ▶ Update $\theta \leftarrow \theta + \alpha \widehat{\nabla_\theta J(\theta)}$

(one may use $N = 1$, and T large enough so that $\gamma^T / (1 - \gamma)$ is small)

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(one may use $N = 1$, and T large enough so that $\gamma^T / (1 - \gamma)$ is small)

Choosing the policy class

A common choice when \mathcal{A} is finite is a **softmax policy**

$$\forall a \in \mathcal{A}, \pi_{\theta}(a|s) = \frac{\exp(\kappa f_{\theta}(s, a))}{\sum_{a' \in \mathcal{A}} \exp(\kappa f_{\theta}(s, a'))}$$

- ▶ if \mathcal{S} is finite, one may use $f_{\theta}(s, a) = \theta_{s,a}$ $\Theta = \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$
- ▶ otherwise, $f_{\theta}(s, a)$ is a function in some parametric space (e.g. a neural network)

$$\nabla_{\theta} \log \pi_{\theta}(a|s) = \kappa \nabla_{\theta} f_{\theta}(s, a) - \kappa \sum_{a' \in \mathcal{A}} \pi_{\theta}(a'|s) \nabla_{\theta} f_{\theta}(s, a')$$

Choosing the policy class

Policy gradient algorithms permit to handle **continuous action spaces** as well. For example, we may use a **Gaussian policy** with density

$$\pi_{\theta}(a|s) = \frac{1}{\sqrt{2\pi\sigma_{\theta_2}^2(s)}} \exp\left(-\frac{(a - \mu_{\theta_1}(s))^2}{2\sigma_{\theta_2}^2(s)}\right)$$

$$\nabla_{\theta_1} \log \pi(a|s) = \frac{(a - \mu_{\theta_1}(s))}{\sigma_{\theta_2}^2(s)} \nabla_{\theta_1} \mu_{\theta_1}(s)$$

$$\nabla_{\theta_2} \log \pi(a|s) = \frac{(a - \mu_{\theta_1}(s))^2 - \sigma_{\theta_2}^2(s)}{\sigma_{\theta_2}^3(s)} \nabla_{\theta_2} \sigma_{\theta_2}(s)$$

Limitation

The gradient estimated by REINFORCE can have a large **variance**

Two ideas to overcome this problem :

- ▶ use better baselines
- ▶ use a different estimate of $Q^{\pi_\theta}(s, a)$
(which will create **biais**)

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Baseline trick, reloaded

One can further subtract the baseline $b(s_1, a_1, \dots, s_t) = V^{\pi_\theta}(s_t)$:

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} Q^{\pi_\theta}(s_t, a_t) \nabla_\theta \log \pi_\theta(s_t | a_t) \right] \\ &= \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} (Q^{\pi_\theta}(s_t, a_t) - V^{\pi_\theta}(s_t)) \nabla_\theta \log \pi_\theta(s_t | a_t) \right] \\ &= \mathbb{E}^{\pi_\theta} \left[\sum_{t=1}^{\infty} \gamma^{t-1} A^{\pi_\theta}(s_t, a_t) \nabla_\theta \log \pi_\theta(s_t | a_t) \right]\end{aligned}$$

introducing the **advantage function**

$$\begin{aligned}A^\pi(s, a) &= Q^\pi(s, a) - V^\pi(s) \\ &= Q^\pi(s, a) - Q^\pi(s, \pi(s))\end{aligned}$$

(how good it is to replace the first action by a when following π ?)

Estimating the advantage

- ▶ Assume we have access to \hat{V} , an estimate of V^{π_θ}
- ▶ The advantage function in (s_t, a_t) can be estimated using the next transition by

$$\hat{A}(s_t, a_t) = r_t + \gamma \hat{V}(s_{t+1}) - \hat{V}(s_t)$$

or more transitions

$$\hat{A}(s_t, a_t) = \sum_{k=t}^{t+p} \gamma^{k-t} r_k + \gamma^{p+1} \hat{V}(s_{t+p+1}) - \hat{V}(s_t)$$

- ▶ This leads to a gradient estimator from (multiple) trajectories

$$\widehat{\nabla_\theta J(\theta)} = \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \gamma^{t-1} \hat{A}(s_t^{(i)}, a_t^{(i)}) \nabla_\theta \log \pi_\theta(a_t^{(i)} | s_t^{(i)})$$

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$$\widehat{\nabla_{\theta} J(\theta)} = \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \gamma^{t-1} \hat{A}(s_t^{(i)}, a_t^{(i)}) \nabla_{\theta} \log \pi_{\theta}(a_t^{(i)} | s_t^{(i)})$$

- How do we produce the estimates \hat{V} ? Use a **critic**

Actor critic algorithms

- ▶ **Actor** : maintains a **policy** and performs trajectory under it
- ▶ **Critic** : maintain a **value**, which estimates the value of the policy followed by the critic

Rationale :

- ▶ the actor's policy *improves* the value given by the critic
- ▶ the critic uses the trajectories generated by the actor to update its *evaluation* of the value
- Generalized Policy Iteration

Both the actor and the critic can use **parametric representation** :

- ▶ π_θ : the actor's policy, $\theta \in \Theta$
- ▶ V_ω : the critic's value, $\omega \in \Omega$

How to update the critic ?

- ▶ **Idea 1** : use TD(0)

after each observed transition under π_θ ,

$$\begin{aligned}\delta_t &= r_t + \gamma V_\omega(s_{t+1}) - V_\omega(s_t) \\ \omega &\leftarrow \omega + \alpha \delta_t \nabla_\omega V_\omega(s_t)\end{aligned}$$

- ▶ **Idea 2** : use batches and bootstrapping

$$\hat{V}(s_t^{(i)}) = \sum_{k=t}^{t+p} \gamma^{k-t} r_k + \gamma^{p+1} V_\omega(s_{t+p+1}^{(i)})$$

and minimize the loss with respect to ω :

$$\frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \left(\hat{V}(s_t^{(i)}) - V_\omega(s_t^{(i)}) \right)^2$$

The A2C algorithm

[Mnih et al., 2016]

In each iteration :

- ▶ collect M transitions under the policy π_θ (with reset of initial states if a terminal state is reached) $\{(s_k, a_k, r_k, s_{k+1})\}_{k \in [M]}$
- ▶ compute the (bootstrap) Monte-Carlo estimate

$$\hat{V}(s_k) = \hat{Q}(s_k, a_k) = \sum_{t=K}^{\tau_k \wedge M} \gamma^{t-k} r_t + \gamma^{M-k+1} V_\omega(s_{M+1}) \mathbb{1}(\tau_k > M)$$

and advantage estimates $\hat{A}_\omega(s_k, a_k) = \hat{Q}(s_k, a_k) - V_\omega(s_k)$.

- ▶ one gradient step to minimize the policy loss : $\theta \leftarrow \theta + \alpha \nabla_\theta L_\pi(\theta)$

$$L_\pi(\omega) = -\frac{1}{M} \sum_{k=1}^M A_\omega(s_k, a_k) \log \pi_\theta(a_k | s_k) - \frac{\gamma}{M} \sum_{k=1}^M \sum_a \pi_\theta(a | s_k) \log \frac{1}{\pi_\theta(a | s_k)}$$

- ▶ one gradient step to minimize the value loss : $\omega \leftarrow \omega + \alpha \nabla_\omega L_V(\omega)$

$$L_V(\omega) = \frac{1}{M} \sum_{k=1}^M \left(\hat{V}(s_k) - V_\omega(s_k) \right)^2$$

Policy Gradient Algorithms :

Pros and Cons

- + allows conservative policy updates (not just taking argmax), which make learning more stable
- + easy to implement and can handle continuous state and action spaces
- + the use of randomized policies allows for some **exploration**...
 - ... but not always enough
 - requires a lot of samples
 - controlling the variance of the gradient can be hard (many tricks for variance reduction)
 - the loss function $J(\theta)$ is *not* concave, how to avoid local maxima?



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