Reinforcement Learning

Lecture 3 : Reinforcement Learning Algorithms

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Reminder : Dynamic Programming

If the parameters of a Markov Decision Process (MDP) are known

- mean reward $(r(s,a))_{(s,a)\in\mathcal{S}\times\mathcal{A}}$
- transition probabilities $(p(s'|s,a))_{(s,a,s')\in\mathcal{S}\times\mathcal{A}\times\mathcal{S}}$

one can compute the optimal value $V^\star$ and optimal policy $\pi^\star$ using the fact that they satisfy the **Bellman equations**.

➜ Finite horizon H : $V_h^\star$ and $\pi_h^\star$ for $h \in \{1,\dots,H\}$ computed using backwards induction from

$$V_h^\star(s) = \max_a \left[r(s,a) + \sum_{s'\in\mathcal{S}} p(s'|s,a) V_{h+1}^\star(s') \right]$$

➜ **Infinite horizon with discount factor** γ (our focus today) : $\pi^\star$ is stationary and

$$V^\star(s) = \max_a \left[r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a) V^\star(s') \right]$$

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➜ **Infinite horizon with discount factor** γ (our focus today) : $\pi^\star$ is stationary and

$$\forall s \in \mathcal{S},\ \ V^\star(s) = T^\star\left(V^\star\right)(s)$$

One may use Value Iteration or Policy Iteration

Reinforcement Learning

- $r(s,a)$ and $p(s'|s,a)$ are unknown, we can only interact with the environment and observe transitions

The RL interaction protocol :

$$\mathcal{H}_t = \sigma\left(s_1, a_1, r_1, s_2, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t\right)$$

denotes the history of observations up to the beginning of round t.

At each time t, the agent

- selects an action $a_t \sim \pi_t(s_t)$ according to some behavior policy

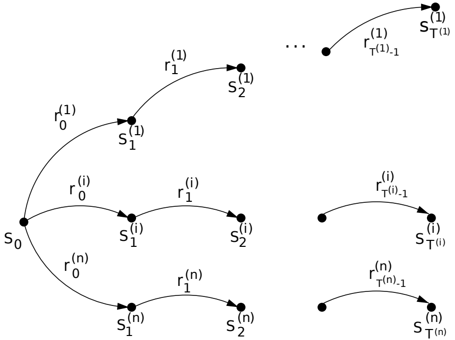
$$\pi_t \quad \text{may depend on } \mathcal{H}_t$$

- observes the reward and next state

$$\begin{cases} r_t & \sim & \nu_{(s_t,a_t)} \text{ such that } \mathbb{E}[r_t|s_t,a_t] = r(s_t,a_t) \\ s_{t+1} & \sim & p(\cdot|s_t,a_t) \end{cases}$$

Reinforcement Learning

For example, starting from some state s_0, one may observe several trajectories under a given policy.



One may also :

- restart in different states
- observe a single, very long, trajectory
- adaptively change the behavior policy

Monte Carlo estimation of a mean

A naive way to estimate a value is to use is definition as an expectation :

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \middle| s_1 = s\right]$$

- Given n (long enough) trajectories under π starting from $s_1^{(i)} = s$,

$$t^{(i)} = (s_1^{(i)}, r_1^{(i)}, s_2^{(i)}, r_2^{(i)}, \ldots, s_{T_{(i)}}^{(i)}, r_{T_{(i)}}^{(i)})$$

one can use the approximation

$$V^\pi(s) \simeq \frac{1}{n} \sum_{i=1}^{n} \underbrace{\left[\sum_{t=1}^{T_{(i)}} \gamma^{t-1} r_t^{(i)}\right]}_{\text{i.i.d. with mean } \simeq V^\pi(s)}.$$

Properties

More generally, considering Z_i that are i.i.d. with mean μ, one can define the Monte-Carlo estimator

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i,$$

which has nice statistical properties, like $\hat{\mu}_n \overset{\text{a.s.}}{\to} \mu$.

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▶ **Iterative rewriting**

$$\hat{\mu}_n = \frac{n-1}{n} \hat{\mu}_{n-1} + \frac{1}{n} Z_n$$

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▶ **Iterative rewriting**

$$\hat{\mu}_n = \hat{\mu}_{n-1} + \alpha_n \left(Z_n - \hat{\mu}_{n-1} \right)$$

for the stepsize $\alpha_n = \frac{1}{n}$.

➜ Can we choose other stepsizes and still have $\hat{\mu}_n \overset{\text{a.s.}}{\to} \mu$?

Stochastic Approximation : Robbins-Monro

Goal : Find the solution to $\phi(x^\star) = 0$ based on access to *noisy* function evaluations, i.e. for every x, one can observe a random value

$$Y = \phi(x) + \varepsilon,$$

where ε has zero mean (conditionally to previous queries).

Robbins-Monro algorithm (1951)

Given an initial x_0, for all $n \geq 1$

- query a noisy evaluation $Y_n = \phi(x_{n-1}) + \varepsilon_n$
- update $x_n = x_{n-1} + \alpha_n Y_n$

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Particular case : estimate a mean μ based on i.i.d. samples Z_i

$$\phi(x) = \mu - x \quad \text{and} \quad Y_n = Z_n - \hat{\mu}_{n-1}$$

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$$\phi(x) = \mu - x \quad \text{and} \quad Y_n = Z_n - \hat{\mu}_{n-1}$$

Robbins-Monro update : $\hat{\mu}_n = \hat{\mu}_{n-1} + \alpha_n(Z_n - \hat{\mu}_{n-1})$.

Convergence of the Robbins-Monro algorithm

Theorem

Let $\phi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$. Under the following assumptions

- ϕ is continuous and $\forall x \neq x^\star$, $(x - x^\star)\phi(x) < 0$
- there exists $C > 0$ such that $\mathbb{E}[Y_n^2 | x_{n-1}] \leq C(1 + x_{n-1}^2)$.
- the stepsizes satisfy

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n^2 < \infty \tag{1}$$

under the Robbins-Monro algorithm, one has $x_n \xrightarrow{a.s} x^\star$.

Consequence : for the mean estimation problem, the sequence of iterates

$$\hat{\mu}_n = \hat{\mu}_{n-1} + \alpha_n (Z_n - \hat{\mu}_{n-1})$$

converges almost surely to μ for any stepsize α_n satisfying (1) if $\mathbb{E}[Z_n^2 | X_{n-1}]$ is finite.

Robbins-Monro for fixed points

Goal : Find the solution to $x^\star = T(x^\star)$ based on access to noisy evaluations of $T(x)$.

Stochastic approximation for a fixed point

Given an initial x_0, for all $n \geq 1$

- query a noisy evaluation Z_n : $\mathbb{E}[Z_n | x_{n-1}] = T(x_{n-1})$.
- update $x_n = x_{n-1} + \alpha_n (Z_n - x_{n-1})$

➜ corresponds to the Robbins-Monro algorithm with

$$\phi(x) = T(x) - x \quad \text{and} \quad Y_n = Z_n - x_{n-1}.$$

Temporal Differences

Given a policy π, we want to compute V^π, which satisfies

$$V^\pi = T^\pi(V^\pi)$$

where $T^\pi(V)(s) = r(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) V(s')$.

- Given a current estimate $\hat{V}$, if we generate a trajectory under π

  $$s_1, r_1, s_2, r_2, \dots, s_T, r_T,$$

  one can produce noisy evaluations of $T^\pi(\hat{V})(s_k)$ for all $k \in \{1, \dots, T-1\}$ using

  $$Z_k = r_k + \gamma \hat{V}(s_{k+1}).$$

$$\mathbb{E}[Z_k | \hat{V}, s_1, r_1, \dots, s_k] = r(s_k, \pi(s_k)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s_k, \pi(s_k)) \hat{V}(s')$$

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- "Robbins-Monro" update : $\hat{V}(s_k) \leftarrow \hat{V}(s_k) + \alpha \left(Z_k - \hat{V}(s_k) \right)$

Temporal Differences

Definition

The Robbins-Monro update rewrites

$$\hat{V}(s_k) \leftarrow \hat{V}(s_k) + \alpha \delta_k(\hat{V})$$

introducing the k-th temporal difference (or TD error) :

$$\delta_k(\hat{V}) := r_k + \gamma \hat{V}(s_{k+1}) - \hat{V}(s_k).$$

- **Interpretation** :

$$\delta_k(\hat{V}) := \underbrace{r_k + \gamma \hat{V}(s_{k+1})}_{\text{new estimate}} - \underbrace{\hat{V}(s_k)}_{\text{previous estimate}}$$

The value of the estimate is moved toward the value of the new estimate, which is itself built upon $\hat{V}$.

➜ Bootstrapping !

Sutton, *Learning to Predict by the Method of Temporal Differences*, 1988

The TD(0) algorithm

Input : π : policy, T : number of iterations, $(\alpha_i(s))_{i\in\mathbb{N}}$: stepsizes,
$V_0 \in \mathbb{R}^S$: initial values, $s_0 \in \mathcal{S}$: initial state (arbitrary)

$V \leftarrow V_0$, $s \leftarrow s_0$
$N \leftarrow 0_S$
for $t = 1, \dots, T$ **do**
  $N(s) \leftarrow N(s) + 1$ \\\\ *update the number of visits of state s*
  $(r, s') = \mathrm{step}(s, \pi(s))$ \\\\ *perform a transition under π*
  $V(s) \leftarrow V(s) + \alpha_{N(s)}(s)\left(r + \gamma V(s') - V(s)\right)$
  $s \leftarrow s'$
end
Return: V

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$$(r, s') = \mathrm{step}(s, \pi(s)) \quad \Leftrightarrow \quad \begin{cases} r & \sim \quad \nu_{(s,\pi(s))} \\ s' & \sim \quad p(\cdot | s, \pi(s)) \end{cases}$$

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➜ tuning the stepsizes ?

The TD(0) algorithm

Theorem

If the step-size (also called *learning rate*) satisfy the Robbins-Monro conditions in all state s :

$$\sum_{i=1}^{\infty} \alpha_i(s) = +\infty \quad \text{and} \quad \sum_{i=1}^{\infty} (\alpha_i(s))^2 < +\infty$$

and *all states are visited infinitely often*, then

$$\lim_{T\to\infty} \hat{V}_T = V^\pi,$$

where $\hat{V}_T$ denotes the output of TD(0) after T iterations.

- **Typical choice :** $\alpha_i(s) = \frac{1}{i^\beta}$ for $\beta \in (1/2, 1]$.

$$\hat{V}_t(s) = \hat{V}_{t-1}(s) + \frac{1}{N_t(s)^\beta}\left(r + \gamma \hat{V}_{t-1}(s') - \hat{V}_{t-1}(s)\right)$$

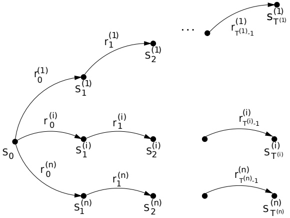
with $N_t(s)$ the number of visits of s up to the t-th iteration.

Monte-Carlo with Temporal Differences

Incremental Monte-Carlo for the estimation of

$$V^{\pi}(s_1) = \mathbb{E}^{\pi}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1\right]$$

based on n trajectories starting in s_1 :



Update after the i-th trajectory :

$$\hat{V}_i(s_1) = \hat{V}_{i-1}(s_1) + \alpha_i \left(\sum_{t=1}^{T^{(i)}} \gamma^{t-1} r_t^{(i)} - \hat{V}_{i-1}(s_1)\right)$$

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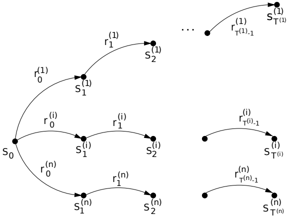


Update after the i-th trajectory : $\rightarrow$ rewrites with the temporal differences

$$\hat{V}_i(s_1) = \hat{V}_{i-1}(s_1) + \alpha_i \left(\sum_{t=1}^{T^{(i)}-1}\gamma^{t-1}\delta_t^{(i)}(\hat{V}_{i-1}) + \gamma^{T^{(i)}-1}\left(r_T^{(i)} - \hat{V}_{i-1}(s_{T^{(i)}})\right)\right)$$

Monte-Carlo with Temporal Differences

$$\hat{V}_i(s_1) \simeq \hat{V}_{i-1}(s_1) + \alpha_i \left(\sum_{t=1}^{T^{(i)}-1} \gamma^t \delta_t^{(i)}(\hat{V}_{i-1}) \right)$$

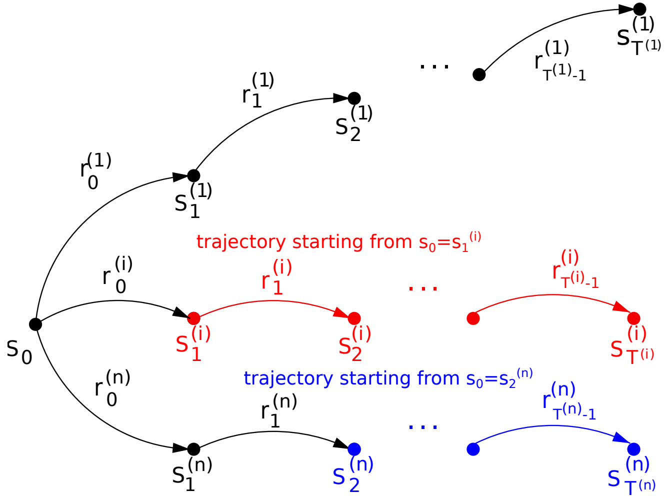
Limitation of naive Monte-Carlo :

- performing a full trajectory is needed before the update
- we only update the value of the initial state s_1

Extension :

- ➜ update the values of multiple states after each trajectory
- ➜ online updates, after each transition

Why update multiple states ?



Every visit Monte-Carlo

Every visits Monte-Carlo (a.k.a. TD(1)) : after the i-th trajectory, instead of updating only $\hat{V}(s_1)$, for all $k = T^{(i)} - 1$ down to 1,

$$\hat{V}\left(s_k^{(i)}\right) \leftarrow \hat{V}\left(s_k^{(i)}\right) + \alpha_i\left(s_k^{(i)}\right)\left(\sum_{t=k}^{T^{(i)}} \gamma^{t-k} r_t^{(i)} - \hat{V}\left(s_k^{(i)}\right)\right)$$

Remarks :

- multiple updates of states visited more than once in the trajectory
- **first visit** variant : update $s_k^{(i)}$ only is $s_k^{(i)} \notin \{s_1^{(i)}, \dots, s_{k-1}^{(i)}\}$

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TD methods for learning the optimal policy ?

TD methods permit to approximately compute V^π for a given policy π

➜ can we use them to get to $\pi^\star$?

Hope : policy evaluation is a central ingredient in **Policy Iteration**

$$\boxed{\pi_0 \to V^{\pi_0} \to \pi_1 = \mathtt{greedy}(V^{\pi_0}) \to V^{\pi_1} \to \pi_2 = \mathtt{greedy}(V^{\pi_1}) \to V^{\pi_2} \to \cdots \to \pi^\star}$$

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Limitation : the policy improvement step cannot be performed without the knowledge of the MDP parameters

$$\begin{aligned} \pi_{k+1} &= \text{greedy}\left(V^{\pi_k}\right) \\ \Leftrightarrow \quad \pi_{k+1}(s) &= \operatorname*{argmax}_{a \in \mathcal{A}} \left[r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^{\pi_k}(s') \right] \end{aligned}$$

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Other possibility : work directly with Q-values !

Reminder : Q-values

$$Q^\pi(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V^\pi(s')$$
$$Q^\star(s,a) = \max_\pi Q^\pi(s,a)$$

Properties

1. $Q^\star$ statisfies the Bellman equations

$$Q^\star(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) \max_{a' \in \mathcal{A}} Q^\star(s',a')$$

2. $V^\star(s) = Q^\star(s, \pi^\star(s))$
3. $\pi^\star = \texttt{greedy}(Q^\star)$, i.e. $\pi^\star(s) = \mathrm{argmax}_{a \in \mathcal{A}} Q^\star(s,a)$

➜ New goal : Learning $Q^\star$

A stochastic approximation scheme for $Q^\star$

- $Q^\star$ also satisfies a fixed point equation : $Q^\star = T^\star(Q^\star)$ where
$$T^\star(Q)(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) \max_{a' \in \mathcal{A}} Q(s',a').$$

- Noisy evaluations of $T^\star(Q)(s_k, a_k)$ along a trajectory :
$$Z_k = r_k + \gamma \max_{a' \in \mathcal{A}} Q(s_{k+1}, a')$$

satisfies $\mathbb{E}[Z_k|\mathcal{H}_k, a_k] = T^\star(Q)(s_k, a_k)$.

(*for any behavior policy*)

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➜ **Robbins-Monro update :**

$$\hat{Q}(s_k, a_k) \leftarrow \hat{Q}(s_k, a_k) + \alpha \left(r_k + \gamma \max_{a' \in \mathcal{A}} \hat{Q}(s_{k+1}, a') - \hat{Q}(s_k, a_k) \right)$$

Q-Learning

Input : T : number of iterations, $(\alpha_i(s,a))_{i\in\mathbb{N}}$: step-sizes,
$Q_0 \in \mathbb{R}^{S\times A}$: initial Q-values, $s_0 \in \mathcal{S}$: initial state (arbitrary)
π_t : behavior policy

$Q \leftarrow Q_0$, $s \leftarrow s_0$
$N \leftarrow 0_{S\times A}$
for $t = 1, \ldots, T$ **do**
  $a \sim \pi_t(s)$ \\\\ *choose an action under the behavior policy*
  $N(s,a) \leftarrow N(s,a) + 1$ \\\\ *update the number of visits of (s,a)*
  $(r, s') = \mathrm{step}(s,a)$ \\\\ *perform a transition*
  $Q(s,a) \leftarrow Q(s,a) + \alpha_{N(s,a)}(s,a)\left(r + \gamma \max_b Q(s',b) - Q(s,a)\right)$
  $s \leftarrow s'$
end
Return: $Q, \pi = \mathtt{greedy}(Q)$

[Watkins, 1989]

Q-Learning

Input : T : number of iterations, $(\alpha_i(s,a))_{i\in\mathbb{N}}$: step-sizes,
$Q_0 \in \mathbb{R}^{S\times A}$: initial Q-values, $s_0 \in \mathcal{S}$: initial state (arbitrary)
π_t : behavior policy

```
Q ← Q_0, s ← s_0
N ← 0_{S×A}
for t = 1, ..., T do
    a ~ π_t(s)                          \\ choose an action under the behavior policy
    N(s,a) ← N(s,a) + 1                 \\ update the number of visits of (s,a)
    (r, s') = step(s,a)                 \\ perform a transition
    Q(s,a) ← Q(s,a) + α_{N(s,a)}(s,a) (r + γ max_b Q(s',b) − Q(s,a))
    s ← s'
end
Return: Q, π = greedy(Q)
```

$$Q \leftarrow Q_0,\; s \leftarrow s_0$$
$$N \leftarrow 0_{S\times A}$$
for $t = 1,\dots,T$ **do**
- $a \sim \pi_t(s)$ \\\\ *choose an action under the behavior policy*
- $N(s,a) \leftarrow N(s,a) + 1$ \\\\ *update the number of visits of (s,a)*
- $(r, s') = \mathrm{step}(s,a)$ \\\\ *perform a transition*
- $Q(s,a) \leftarrow Q(s,a) + \alpha_{N(s,a)}(s,a)\left(r + \gamma \max_b Q(s',b) - Q(s,a)\right)$
- $s \leftarrow s'$

end

Return: $Q, \pi = \mathtt{greedy}(Q)$

[Watkins, 1989]

Q-Learning

Theorem

It the step-size (also called *learning rate*) satisfy the Robbins-Monro conditions in all state action pair (s, a) :

$$\sum_{i=1}^{\infty} \alpha_i(s,a) = +\infty \quad \text{and} \quad \sum_{i=1}^{\infty} (\alpha_i(s,a))^2 < +\infty$$

and *all states-action pairs are visited infinitely often* , then

$$\lim_{T \to \infty} \hat{Q}_T = Q^\star,$$

where $\hat{Q}_T$ denotes the output of T iterations of Q-Learning.

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where $\hat{Q}_T$ denotes the output of T iterations of Q-Learning.

➜ typical step-sizes choice : $\alpha_i(s,a) = \frac{1}{i^\beta}$ with $\beta \in (1/2, 1]$.

Behavior Policy

- **Constraint** : all state-action pairs need to be visited infinitely often

$$\pi_t(s) = \mathcal{U}(\mathcal{A}) \;\rightarrow\; a_t \text{ chosen uniformly at random?}$$

- **Idea** : we care about $\pi^\star$, we need to refine our estimate of $Q^\star$ in the pairs $(s, \pi^\star(s))$ / we may want to maximize rewards while learning

$$\pi_t = \texttt{greedy}\left(\hat{Q}_{t-1}\right)?$$

ε-greedy exploration [Sutton and Barto, 2018]

The **ε-greedy** policy performs the following :

➜ with probability ε, select $a_t \sim \mathcal{U}(\mathcal{A})$

➜ with probability $1-\varepsilon$, select $a_t = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ \hat{Q}_t(s_t, a)$

➜ tends to the greedy policy when $\varepsilon \rightarrow 0$

Behavior Policy

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Boltzmann (or softmax) exploration [Sutton and Barto, 2018]

The **softmax policy** with temperature τ is given by

$$(\pi_t(s))_a = \frac{\exp(\hat{Q}_t(s,a)/\tau)}{\sum_{a' \in \mathcal{A}} \exp(\hat{Q}_t(s,a')/\tau)}$$

and $a_t \sim \pi_t(s_t)$.

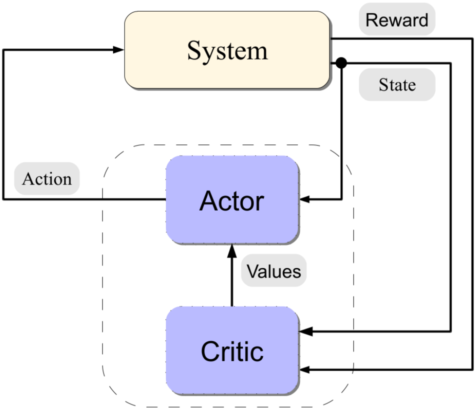
➜ tends to the greedy policy when $\tau \rightarrow 0$

In practice

- Q-Learning (and more generaly TD methods) can be very slow to converge...

The Actor/Critic architecture

- **the actor** : update its policy to improve the value given by the critic
- **the critic** : evaluates the actor's policy



source : [Szepesvári, 2010]

Generalized Policy Iteration

Policy Iteration is an extreme example of an Actor/Critic architecture :

- **the actor** : "acts" with $\pi = \texttt{greedy}(V)$ where V is the value provided by the critic
- **the critic** : computes V^π where π is the current actor's policy

Generalized Policy Iteration

Policy Iteration is an extreme example of an Actor/Critic architecture :

- **the actor** : performs policy improvement
- **the critic** : performs policy evaluation

➜ Actor/Critic is also referred to as **Generalized Policy Iteration**

[Sutton and Barto, 2018]

There are many algorithms of this type !

An example : the SARSA algorithm

- **The critic**

After observing the actor's recent behavior $(s_t, a_t, r_t, s_{t+1}, a_{t+1})$, update

$$\hat{Q}(s_t, a_t) \leftarrow \hat{Q}(s_t, a_t) + \alpha \left(r_t + \gamma \hat{Q}(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t) \right)$$

State Action Reward State Action (SARSA) update

➜ if the actor is following a fixed policy π ($a_t = \pi(s_t)$), SARSA=TD(0)

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State Action Reward State Action (SARSA) update

➜ if the actor is following a fixed policy π ($a_t = \pi(s_t)$), SARSA=TD(0)

- **The actor** : moves its behavior policy towards being greedy with respect to the Q-value provided by the critic, e.g.
  - ➜ ε-greedy policy
  - ➜ softmax policy with temperature τ

Q-Learning versus SARSA

The update rules of the two algorithms are close but not identical :

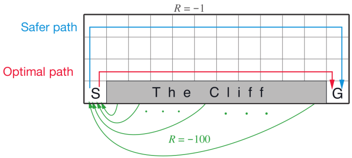
- **Q-Learning :**
$$\hat{Q}(s_t, a_t) \leftarrow \hat{Q}(s_t, a_t) + \alpha \left(r_t + \gamma \max_{a'} \hat{Q}(s_{t+1}, a') - \hat{Q}(s_t, a_t) \right)$$
- **SARSA :**
$$\hat{Q}(s_t, a_t) \leftarrow \hat{Q}(s_t, a_t) + \alpha \left(r_t + \gamma \hat{Q}(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t) \right)$$

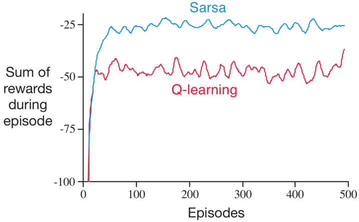
Both aim at learning the target policy $\pi^\star(s) = \mathrm{argmax}_a Q^\star(s, a)$.

- Q-Learning converges for any behavior policy (exploring enough)
off-policy learning
- for SARSA the bahavior policy is close to the estimated target policy
on-policy learning

Q-Learning versus SARSA

An example from [Sutton and Barto, 2018] : Q-Learning and SARSA used with ε-greedy exploration with $\varepsilon = 0.1$.





Observation : SARSA converges to a sub-optimal safer policy that yield more reward during learning, while Q-Learning converges to the optimal policy, while falling often from the cliff during learning

(if $\varepsilon \to 0$, SARSA would also converge to the optimal policy)

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