Sequential Decision Making

Lecture 4 : Reinforcement Learning with Function Approximation

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Different Markov Decision Problems

Overall goal : learn the optimal policy $\pi^\star$ associated to some MDP parameterized by $r(s, a)$ and $p(\cdot|s, a)$ for $(s, a) \in \mathcal{S} \times \mathcal{A}$.

Different contexts :

1. Small state space $\mathcal{S}$, known dynamics
2. Small state space $\mathcal{S}$, unknown dynamics
3. Large state space $\mathcal{S}$, known dynamics
4. Large state space $\mathcal{S}$, unknown dynamics

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   Q-Learning. SARSA.
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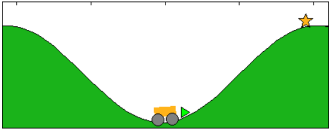
Example : Mountain Car



State : $(x, \dot{x}) \in [-1.2; 0.6] \times [-0.07; 0.07]$

Actions : $\mathcal{A} = \{-1, 0, 1\}$:
full speed backwards / do nothing / full speed forward

Reward : always -1 except in the terminal (goal) state $x_\star = 0.6$

Dynamics : when doing action a_t in state $s_t = (x_t, v_t)$, the next state $s_{t+1} = (x_{t+1}, v_{t+1})$ is

$$\begin{cases} v_{t+1} & = & \max\{\min\{v_t + \epsilon_t + 0.001 a_t - 0.0025 \cos(3x_t), 0.07\}, -0.07\}, \\ x_{t+1} & = & \max\{\min\{x_t + v_t, 0.6\}, -1.2\}. \end{cases}$$

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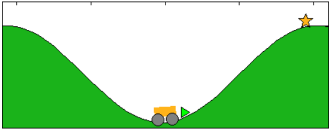

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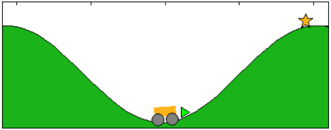

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➜ for others, this is a "continuous space, unknown dynamics" setting

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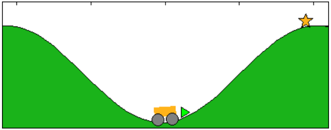
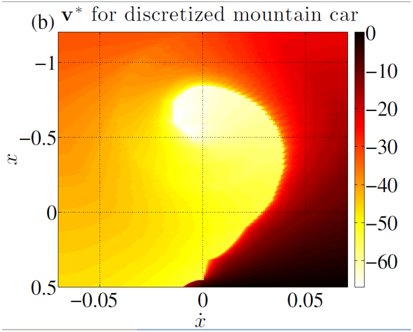


The optimal policy is to first climb up the other side :



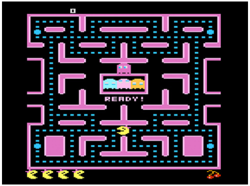
More “Large space, Unknown Dynamics”

Many concrete problems where RL could be applied fall in this framework

- micro-grid management
- self-driving cars
- autonomous robotics ...

Benchmarks often used by researcher these days are video games :

- ➜ dynamics may be unknown (enemies behavior, random level generation...)
- ➜ state-space may be large (e.g., pixels)



Outline

Learning Values or Q-Values

In RL, one often learn values instead of policy directly :

$$V^\star(s) = \max_\pi \mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \middle| s_1 = s \right]$$

Property : $V^\star(s) = \max_a Q^\star(s,a)$.

From an estimate of $V^\star$ to an estimate of $Q^\star$

$$Q \xrightarrow{\text{easy}} V(s) = \max_a Q(s,a)$$

$$V \xrightarrow{\text{possibly harder}} Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)}[V(s)]$$

The policy deduced from an estimate V is $\pi = \text{greedy}(V)$

$$\pi(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \left(r(s,a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)}[V(s')] \right)$$

➜ decide when to approximate $V^\star$ or $Q^\star$

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$$V \xrightarrow{\text{possibly harder}} Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)}[V(s)]$$

The policy deduced from an estimate Q is $\pi = \text{greedy}(Q)$

$$\pi(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \; Q(s,a)$$

➜ decide when to approximate $V^\star$ or $Q^\star$

From Values to Policies

Question : how does the approximation error $\|V - V^\star\|$ impact the performance loss of the policy deduced from V ?

Proposition

Let V be an approximation of $V^\star$ and $\pi = \text{greedy}(V)$.

$$\underbrace{\|V^\star - V^\pi\|_\infty}_{\text{performance loss}} \leq \frac{2\gamma}{1-\gamma} \underbrace{\|V^\star - V\|_\infty}_{\text{approximation error}} .$$

- also, $\|V^\star - V\|_\infty \leq \|Q^\star - Q\|_\infty$ if $V(s) = \max_a Q(s,a)$.

Exercise : Prove it !

Proof

Ingredients :

- $T^\star V^\star = V^\star$ and $T^\pi V^\pi = V^\pi$
- both $T^\star$ and T^π are γ contractions wrt $\|\cdot\|_\infty$
- as $\pi = \text{greedy}(V)$, $T^\star V = T^\pi V$

$$\begin{aligned}
\|V^\star - V^\pi\|_\infty &\le \|T^\star V^\star - T^\pi V\|_\infty + \|T^\pi V - T^\pi V^\pi\|_\infty \\
&\le \|T^\star V^\star - T^\star V\|_\infty + \gamma\|V - V^\pi\|_\infty \\
&\le \gamma\|V^\star - V\|_\infty + \gamma(\|V - V^\star\|_\infty + \|V^\star - V^\pi\|_\infty)
\end{aligned}$$

Hence

$$\|V^\star - V^\pi\|_\infty \le \frac{2\gamma}{1-\gamma}\|V^\star - V\|_\infty .$$

Value Functions Approximation

Problem : Often $\mathcal{S}$ is too large to store a vector $V \in \mathbb{R}^S$ or a table $Q \in \mathbb{R}^{S \times A}$ in memory...

Solution : look for estimates V (resp. Q) of $V^\star$ (resp. Q^*) in an approximation space $\mathcal{F}_V$ (resp. $\mathcal{F}_Q$)

$$\mathcal{F}_V \subseteq \mathcal{F}(\mathcal{S}, \mathbb{R}) \qquad \mathcal{F}_Q \subseteq \mathcal{F}(\mathcal{S} \times \mathcal{A}, \mathbb{R})$$

- **Parametric approximation :**

$$\mathcal{F}_V = \left\{ s \mapsto V_\theta(s) \;\middle|\; \theta \in \Theta \right\} \quad \mathcal{F}_Q = \left\{ (s,a) \mapsto Q_\theta(s,a) \;\middle|\; \theta \in \Theta \right\}$$

➜ only requires to store a parameter θ (typically in $\mathbb{R}^d$ with $d \ll |\mathcal{S}|$)

Smooth parameterization if $\nabla_\theta V_\theta(s)$ (resp. $\nabla_\theta Q_\theta(s,a)$) can be computed

Value Functions Approximation

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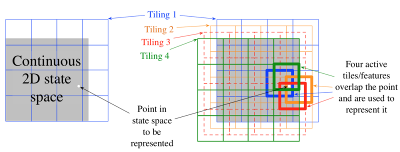
- **Non-parametric approximation :**
  - ➜ nearest neighbors
  - ➜ kernel smoothing

$$V_n(s) = \sum_{t=1}^{n} v_t \frac{K(x, s_t)}{\sum_{\ell=1}^{n} K(x, s_\ell)} \quad \text{for some kernel } K$$

  - ➜ tile coding



Linear function approximation

V is some linear combinations of *basis functions* (or *features*).

$$\mathcal{F}_V = \left\{ s \mapsto V_\theta(s) = \sum_{i=1}^{d} \theta_i \phi_i(s) \;\middle|\; \theta \in \mathbb{R}^d \right\}$$

Introducing the feature vector of a state s

$$\phi(s) = (\phi_1(s), \dots, \phi_d(s))^\top \in \mathbb{R}^d$$

one can write

$$V_\theta(s) = \theta^\top \phi(s).$$

Remarks :

- smooth parameterization with $\nabla_\theta V_\theta(s) = \phi(s)$
- if $\mathcal{S} = \{s_1, \dots, s_S\}$, one recovers the tabular case with $\phi_i(s) = \mathbb{1}(s = s_i)$ for $i = 1, \dots, S$

Linear function approximation

Q is some linear combinations of *basis functions* (or *features*).

$$\mathcal{F}_Q = \left\{ (s,a) \mapsto Q_\theta(s,a) = \sum_{i=1}^{d} \theta_i \phi_i(s,a) \;\middle|\; \theta \in \mathbb{R}^d \right\}$$

Introducing the feature vector of a state-action pair (s,a)

$$\phi(s,a) = (\phi_1(s,a), \dots, \phi_d(s,a))^\top \in \mathbb{R}^d$$

one can write

$$Q_\theta(s,a) = \theta^\top \phi(s,a).$$

Remarks :

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- if $\mathcal{S} = \{s_1, \dots, s_S\}$, $\mathcal{A} = \{a_1, \dots, a_A\}$ one recovers the tabular case with $\phi_{i,j}(s,a) = \mathbb{1}(s = s_i, a = a_j)$ for $i = 1, \dots, S$ and $j = 1, \dots, A$

Examples of features

- $\mathcal{S} \subseteq \mathbb{R}$: one may use polynomial or Fourrier basis
- $\mathcal{S} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_K$: one may use tensor products of features

$$\phi_{(i_1,\dots,i_K)}\left(\left(s^{(1)},\dots,s^{(K)}\right)\right) = \prod_{j=1}^{K} \phi_{i_j}\left(s^{(j)}\right)$$

RBF features

If $\mathcal{S} \subseteq \mathbb{R}^d$, one can use Radial Basic Functions

$$\phi_i(s) = \exp\left(-\eta \|s - s^{(i)}\|^2\right),$$

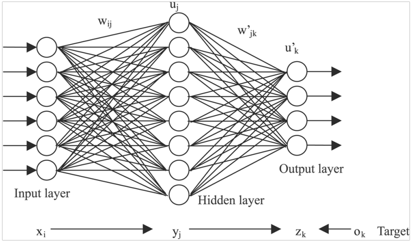
with some scale parameter η and “centers” $s^{(1)},\dots,s^{(d)}$ (e.g. a uniform covering of $\mathcal{S}$, or random centers)

Non linear function approximation

Linear function approximation requires to design (meaningful) features, which can be hard...

Modeling V as a neural network can be more powerful :

- neural networks are known to be universal approximators
- they "learn features" from the data
- and $\nabla_\theta V_\theta(s)$ can still be computed efficiently



Outline

Performance measure

In the tabular case, we proposed algorithms that converge to the **exact** V^π. This is in general hopeless with function approximation.

➜ we can instead try to minimize the Mean Square Error

Mean Square Value Error

Let ν be some probability measure on the state space $\mathcal{S}$ and $V : \mathcal{S} \to \mathbb{R}$.

$$\texttt{MSVE}_\nu(V) = \mathbb{E}_{s\sim\nu}\left[\left(V^\pi(s) - V(s)\right)^2\right]$$

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- ➜ what measure ν do we choose ?

Assumption. Under the policy π, the sequence of visited state $(s_t)_{t\in\mathbb{N}}$ is a Markov chain. We assume that it admits a stationary distribution ν.

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Remark : defining $||\cdot||_\nu$ to be the norm associated to the scalar product

$$\langle f|g\rangle_\nu = \mathbb{E}_{s\sim\nu}\left[f(s)g(s)\right],$$

one has

$$\texttt{MSVE}_\nu(V) = ||V^\pi - V||_\nu^2$$

Minimizing the $\texttt{MSVE}$

We consider a smooth parametric representation for V, $\mathcal{F} = \{V_\theta, \theta \in \Theta\}$, for which we can define

$$\texttt{MSVE}(\theta) = \mathbb{E}_\nu \left[\left(V^\pi(s) - V_\theta(s) \right)^2 \right]$$

and we aim for $\theta^\star = \mathrm{argmin}_{\theta \in \Theta} \ \texttt{MSVE}(\theta)$.

Given the smooth parameterization, one can compute

$$\nabla_\theta \texttt{MSVE}(\theta) = -2\mathbb{E}_\nu \left[\left(V^\pi(s) - V_\theta(s) \right) \nabla_\theta V_\theta(s) \right]$$

(*valid for finite state space, and possibly under some assumption in continuous state spaces*)

Gradient descent :

$$\theta_t \leftarrow \theta_{t-1} + \alpha_t \times \mathbb{E}_\nu \left[\left(V^\pi(s) - V_{\theta_{t-1}}(s) \right) \nabla_\theta V_{\theta_{t-1}}(s) \right]$$

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Stochastic gradient descent :

$$\theta_t \leftarrow \theta_{t-1} + \alpha_t \times \left(V^\pi(s_t) - V_{\theta_{t-1}}(s_t) \right) \nabla_\theta V_{\theta_{t-1}}(s_t)$$

(for large t, s_t is approximately distributed under ν)

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➜ problem : $V^\pi(s_t)$ is unknown...

A semi-gradient approach

Idea : in the stochastic gradient descent update

$$\theta_t \leftarrow \theta_{t-1} + \alpha_t \times \left(V^\pi(s_t) - V_{\theta_{t-1}}(s_t) \right) \nabla_\theta V_{\theta_{t-1}}(s_t)$$

replace $V^\pi(s_t)$ by either

- a Monte-Carlo estimate (TD(1))
- a "Bootstrap" estimate (TD(0))

TD(0) with smooth function approximation

The TD(0) **semi-gradient** update is

$$\theta_t \leftarrow \theta_{t-1} + \alpha_t \times \left(r_t + \gamma V_{\theta_{t-1}}(s_{t+1}) - V_{\theta_{t-1}}(s_t) \right) \nabla_\theta V_{\theta_{t-1}}(s_t)$$

⚠ this is *not* a stochastic gradient update, hence the terminology

➜ stepsize tuning : decaying not too fast (Robbins-Monro style)

➜ very few convergence guarantees besides the **linear case**...

TD(0) with linear function approximation

We assume $V_\theta(s) = \theta^\top \phi(s)$ with the feature vector

$$\phi(s) = (\phi_1(s), \dots, \phi_d(s))^\top \in \mathbb{R}^d.$$

Then $\nabla_\theta V_\theta(s) = \phi(s)$ and the algorithm becomes

TD(0) with linear function approximation

Along a trajectory following π, after observing (s_t, r_t, s_{t+1}) update

$$\theta_t = \theta_{t-1} + \alpha_t \left(r_t + \gamma\theta_{t-1}^\top \phi(s_{t+1}) - \theta_{t-1}^\top \phi(s_t)\right) \phi(s_t).$$

Using the notation $\phi_t = \phi(s_t)$, one has

$$\theta_t = \theta_{t-1} + \alpha_t \left(r_t\phi_t - \phi_t(\phi_t - \gamma\phi_{t+1})^\top \theta_{t-1}\right).$$

Convergence properties

Theorem

Under the following assumptions :

1. the Markov chain $(s_t)_{t\in\mathbb{N}}$ admits a stationary distribution ν
2. the state space is finite and the vectors $\phi_i = (\phi_i(s))_{s\in\mathcal{S}} \in \mathbb{R}^S$ are linearly independent
3. the step-sizes satisfy the Robbins-Monro conditions, i.e.

$$\sum_{t=1}^{\infty} \alpha_t = \infty \quad \text{and} \quad \sum_{t=1}^{\infty} \alpha_t < \infty$$

then the parameter θ_t converges almost surely to some value θ_{TD} s.t.

$$V_{\theta_{\text{TD}}} = \underbrace{\Pi_{\mathcal{F},\nu} T^{\pi}}_{\substack{\text{projected} \\ \text{Bellman operator}}} V_{\theta_{\text{TD}}}$$

$$\Pi_{\mathcal{F},\nu} T^{\pi}(V) = \operatorname*{argmin}_{f\in\mathcal{F}} ||T^{\pi}(V) - f||_{\nu}$$

[Tsitsiklis and Van Roy, 1996]

Computing the fixed point

According to the theorem, TD(0) converges to the solution to

$$V_{\theta_{\mathrm{TD}}} = \Pi_{\mathcal{F},\nu} T^{\pi} V_{\theta_{\mathrm{TD}}}$$

Proposition

The vector θ_{TD} can be obtained as a solution to the linear system

$$A^{\pi} \theta_{\mathrm{TD}} = b^{\pi},$$

where

$$\begin{aligned} A^{\pi}_{i,j} &= \langle \phi_i | \phi_j - \gamma P^{\pi} \phi_j \rangle_{\nu} \\ b^{\pi}_i &= \langle r^{\pi} | \phi_i \rangle \end{aligned}$$

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$$\begin{aligned} A^\pi &= \mathbb{E}_{\substack{s\sim\nu \\ s'\sim p(\cdot|s,\pi(s))}} \left[\phi(s)\left(\phi(s) - \gamma\phi(s')\right)^\top\right] \in \mathbb{R}^{d\times d} \\ b^\pi &= \mathbb{E}_{s\sim\nu}\left[r(s,\pi(s))\phi(s)\right] \in \mathbb{R}^d \end{aligned}$$

Why does TD(0) converge to θ_{TD} ?

(heuristic argument in [Sutton and Barto, 2018])

Recall the TD(0) update :

$$\begin{aligned}\theta_t &= \theta_{t-1} + \alpha_t \left(r_t \phi(s_t) - \phi(s_t)(\phi(s_t) - \gamma \phi(s_{t+1}))^\top \theta_{t-1} \right) \\ &= \theta_{t-1} + \alpha_t \left(b_t - A_t \theta_{t-1} \right),\end{aligned}$$

where we introduce

$$\begin{aligned}A_t &= \phi(s_t)(\phi(s_t) - \gamma \phi(s_{t+1}))^\top \in \mathbb{R}^{d \times d} \\ b_t &= r_t \phi(s_t) \in \mathbb{R}^d\end{aligned}$$

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$$\begin{aligned}\theta_t &= \theta_{t-1} + \alpha_t \left(r_t \phi(s_t) - \phi(s_t)(\phi(s_t) - \gamma\phi(s_{t+1}))^\top \theta_{t-1} \right) \\ &= \theta_{t-1} + \alpha_t \left(b_t - A_t \theta_{t-1} \right),\end{aligned}$$

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$$\begin{aligned}A_t &\simeq \mathbb{E}_{s_t \sim \nu} \left[\phi(s_t)(\phi(s_t) - \gamma\phi(s_{t+1}))^\top \right] = A^\pi \\ b_t &\simeq \mathbb{E}_{s_t \sim \nu} \left[r_t \phi(s_t) \right] = b^\pi\end{aligned}$$

when t is large as s_t is approximately drawn under ν.

Why does TD(0) converge to $\theta_{\mathtt{TD}}$?

(heuristic argument in [Sutton and Barto, 2018])

Recall the TD(0) update :

$$\begin{aligned}\theta_t &= \theta_{t-1} + \alpha_t \left(r_t\phi(s_t) - \phi(s_t)(\phi(s_t) - \gamma\phi(s_{t+1}))^\top \theta_{t-1}\right) \\ &= \theta_{t-1} + \alpha_t \left(b_t - A_t\theta_{t-1}\right),\end{aligned}$$

where we introduce

$$\begin{aligned}A_t &\simeq \mathbb{E}_{s_t\sim\nu}\left[\phi(s_t)(\phi(s_t) - \gamma\phi(s_{t+1}))^\top\right] = A^\pi \\ b_t &\simeq \mathbb{E}_{s_t\sim\nu}\left[r_t\phi(s_t)\right] = b^\pi\end{aligned}$$

when t is large as s_t is approximately drawn under ν.

Approximate recursion :

$$\theta_t = \theta_{t-1} + \alpha\left(b^\pi - A^\pi\theta_{t-1}\right)$$

If it converges, the convergence is towards a fixed point, satisfying

$$b^\pi - A^\pi\theta = 0$$

Least Square Temporal Difference

Idea : Now that we know towards what TD(0) converges, is there a way to get there faster ?

$$A^\pi \theta_{\mathrm{TD}} = b^\pi,$$

where

$$\begin{aligned} A^\pi &= \mathbb{E}_{\substack{s\sim\nu \\ s'\sim p(\cdot|s,\pi(s))}}\left[\phi(s)\left(\phi(s) - \gamma\phi(s')\right)^\top\right] \in \mathbb{R}^{d\times d} \\ b^\pi &= \mathbb{E}_{s\sim\nu}\left[r(s,\pi(s))\phi(s)\right] \in \mathbb{R}^d \end{aligned}$$

- use estimation :

$$\hat{A}_n = \frac{1}{n}\sum_{t=1}^{n}\phi(s_t)\left(\phi(s_t) - \gamma\phi(s_{t+1})\right)^\top \quad \text{and} \quad \hat{b}_n = \frac{1}{n}\sum_{t=1}^{n} r_t\phi(s_t)$$

If $\hat{A}_n$ is invertible, $\hat{\theta}_n = \hat{A}_n^{-1}\hat{b}_n$.

An online implementation of LSTD

We need to compute

$$\hat{\theta}_n = \hat{A}_n^{-1}\hat{b}_n$$

where

$$\hat{A}_n = \sum_{t=1}^{n} \phi(s_t)\left(\phi(s_t) - \gamma\phi(s_{t+1})\right)^\top \quad \text{and} \quad \hat{b}_n = \sum_{t=1}^{n} r_t\phi(s_t).$$

➜ requires to invert a $d \times d$ matrix at every round... (much more costly than the TD(0) update !)

More efficient : update the inverse online !

Sherman-Morrison formula

For any matrix $B \in \mathbb{R}^{d\times d}$ and vectors $u, v \in \mathbb{R}^d$,

$$\left(B + uv^\top\right)^{-1} = B^{-1} - \frac{B^{-1}uv^\top B^{-1}}{1 + v^\top B^{-1}u}$$

LSTD update versus TD(0) update

Letting $\phi_t = \phi(s_t)$, both update also rely on temporal differences

$$\delta_t(\theta) = r_t + \gamma \phi_{t+1}^\top \theta - \phi_t^\top \theta$$

Recursive LSTD

$$\begin{aligned} C_n &= C_{n-1} - \frac{C_{n-1}\phi_n(\phi_n - \gamma\phi_{n+1})^\top C_{n-1}}{1 + (\phi_n - \gamma\phi_{n+1})^\top C_{n-1}\phi_n} \\ \theta_n &= \theta_{n-1} + \frac{C_{n-1}}{1 + (\phi_n - \gamma\phi_{n+1})^\top C_{n-1}\phi_n}\delta_n(\theta_{n-1})\phi_n \end{aligned}$$

TD(0)

$$\theta_n = \theta_{n-1} + \alpha_n \delta_n(\theta_{n-1})\phi_n$$

Complexity : $O(d^2)$ versus 0(1) but LSTD converges faster

Wait... How good is the TD solution ?

We presented two algorithms which converge to the value function

$$V_{\text{TD}}(s) = \theta_{\text{TD}}^\top \phi(s)$$

such that V_{TD} is a fixed point to $\Pi_{\mathcal{F},\nu} T^\pi$ (when it exists).

➜ Is it at all close to our target V^π ?

Proposition

If ν is the stationary distribution of the sequence of states

- $\Pi_{\mathcal{F},\nu} T^\pi$ is a γ contraction with respect to $||\cdot||_\nu$ and admits therefore a unique fixed point, V_{TD}
- The TD solution satisfies

$$||V^\pi - V_{\text{TD}}||_\nu \leq \frac{1}{\sqrt{1-\gamma^2}} \inf_{V \in \mathcal{F}} ||V^\pi - V||_\nu$$

Answer : not too far from the best possible approximation (wrt to $||\cdot||_\nu$)

Outline

Reminder : Policy Iteration

1. Let π_0 be *any* stationary policy
2. At each iteration $k = 1, 2, \ldots, K$
   - ➜ Policy evaluation : given π_{k-1}, compute $V_k = V^{\pi_{k-1}}$.
   - ➜ Policy improvement : compute the *greedy* policy

$$\pi_k(s) \in \operatorname*{argmax}_{a \in \mathcal{A}} \big[r(s,a) + \gamma \mathbb{E}_{s' \sim p(\cdot|s,a)}[V_{k-1}(s')] \big].$$

3. Return the last policy π_K

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- **Problem** : we saw how to approximately perform policy evaluation, how about policy improvement ?

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- **Problem** : we saw how to approximately perform policy evaluation, how about policy improvement ?

➜ work with Q-values directly to make policy improvement easy !

LSTD-Q

LSTD-Q : a variant of LSTD aimed at estimating directly Q^π

$$Q_\theta(s,a) = \theta^\top \phi(s,a)$$

The solution to

$$Q_\theta = \Pi_{\mathcal{F},\nu} T^\pi Q_\theta$$

can similarly be approximated by solving a linear system.

$$\begin{cases} A_n &= A_{n-1} + \phi(s_n,a_n)(\phi(s_n,a_n) - \gamma\phi(s_{n+1},\pi(s_{n+1})))^\top \\ b_n &= b_{n-1} + \phi(s_n,a_n) r_n \end{cases}$$

$$\theta_n^{\texttt{LSTD-Q}} = A_n^{-1} b_n$$

➜ The resulting algorithm is **Least-Squares Policy Iteration** (LSPI) [Lagoudakis and Parr, 2003]

Reminder : Value Iteration

1. Let Q_0 be *any* action-value function
2. At each iteration $k = 1, 2, \ldots, K$

$$\begin{aligned} Q_k(s,a) &= T^\star Q_{k-1}(s,a) \\ &= r(s,a) + \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[\max_{a' \in \mathcal{A}} Q_{k-1}(s',a') \right] \end{aligned}$$

3. Return the greedy policy

$$\pi_K(s) \in \operatorname*{argmax}_{a \in \mathcal{A}} Q_K(s,a).$$

Reminder : Value Iteration

1. Let Q_0 be *any* action-value function
2. At each iteration $k = 1, 2, \ldots, K$

$$\begin{aligned} Q_k(s,a) &= T^\star Q_{k-1}(s,a) \\ &= r(s,a) + \mathbb{E}_{s' \sim p(\cdot|s,a)} \left[\max_{a' \in \mathcal{A}} Q_{k-1}(s',a') \right] \end{aligned}$$

3. Return the greedy policy

$$\pi_K(s) \in \operatorname*{argmax}_{a \in \mathcal{A}} Q_K(s,a).$$

➜ **Problem** : how can we approximate $T^\star Q_k$?

➜ **Problem** : does value iteration still work with such an approximation ?

Fitted-Q Iteration

Input : number of iterations K, number of samples per iteration n,
Initial function $Q_0 \in \mathcal{F}$, sampling distribution ρ,
Approximation space $\mathcal{F}$, loss function ℓ

for $k = 1, \ldots, K$ **do**

  Draw n samples $(s_i, a_i) \sim \rho$

  Perform n transitions $r_i, s_i' = \texttt{step}(s_i, a_i)$

  Compute the targets $y_i = r_i + \gamma \max_a Q_{k-1}(s_i', a)$

  From the training dataset $\mathcal{D}_k = \{((s_i, a_i), y_i)_{1 \le i \le n}\}$, solve the empirical risk minimization problem :

$$f \in \operatorname*{argmin}_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell\left(y_i, f(s_i, a_i)\right)$$

  Set $Q_k = f$ (with clipping if $f(s,a) \notin [-\frac{R_{\max}}{1-\gamma}; \frac{R_{\max}}{1-\gamma}]$).

end

Return: $\pi = \texttt{greedy}(Q_K)$

- ERM can be replaced by other possibly non-parameteric regression techniques (decision trees, k-nn, ...)

Linear Fitted Q-Iteration

Input : number of iterations K, number of samples per iteration n, Initial function $Q_0 \in \mathcal{F}$, sampling distribution ρ, Approximation space $\mathcal{F}$, loss function ℓ

for $k = 1, \dots, K$ **do**

  Draw n samples $(s_i, a_i) \sim \rho$

  Perform n transitions $r_i, s_i' = \texttt{step}(s_i, a_i)$

  Compute the targets $y_i = r_i + \gamma \max_a Q_{k-1}(s_i', a)$

  From the training dataset $\mathcal{D}_k = \{((s_i, a_i), y_i)_{1 \le i \le n}\}$, solve the least squares problem :

$$\theta_k \in \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \theta^\top \phi(s_i, a_i)\right)^2$$

  Set $Q_k(s, a) = \theta_k^\top \phi(s, a)$ (with clipping).

end

Return: $\pi = \texttt{greedy}(Q_K)$

Linear Fitted-Q : Sampling

1. Draw n samples $(s_i, a_i) \overset{\text{i.i.d}}{\sim} \rho$
2. Perform a transition for each of them :
   $s_i' \sim p(\cdot|s_i, a_i)$ and $r_i \sim \nu_{(s_i,a_i)}$

Linear Fitted-Q : Sampling

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- In practice sampling can be done once before running the algorithm (or a database of transitions can be used)
- The sampling distribution ρ should cover the state-action space in all *relevant regions*

- The algorithm requires call to a simulator which can simulate *independent transitions* from anywhere in the state-action space

Linear Fitted-Q : Building the training set

3. Compute $y_i = r_i + \gamma \max_a Q_{k-1}(s_i', a)$
4. Build training set $\mathcal{D}_k = \{((s_i, a_i), y_i)_{1 \leq i \leq n}\}$

Linear Fitted-Q : Building the training set

3. Compute $y_i = r_i + \gamma \max_a Q_{k-1}(s_i', a)$
4. Build training set $\mathcal{D}_k = \{((s_i, a_i), y_i)_{1 \leq i \leq n}\}$

➜ Each sample y_i is an unbiased estimate of $T^\star Q_{k-1}(s_i, a_i)$:

$$\begin{aligned}
\mathbb{E}[y_i | s_i, a_i, Q_{k-1}] &= \mathbb{E}[r_i + \gamma \max_{a'} Q_{k-1}(s_i', a') | s_i, a_i, Q_{k-1}] \\
&= r(s_i, a_i) + \gamma \mathbb{E}_{s' \sim p(\cdot | s_i, a_i)}[\max_{a'} Q_{k-1}(s', a')] \\
&= T^\star Q_{k-1}(s_i, a_i)
\end{aligned}$$

➜ The problem "reduces" to standard regression

➜ A new regression problem at each iteration :
new function to fit $T^\star Q_{k-1}$ + new training set $\mathcal{D}_k$

Linear Fitted-Q : The regression problem

5. Solve the least squares problem

$$\theta_k \in \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \theta^\top \phi(s_i, a_i) \right)^2$$

Linear Fitted-Q : The regression problem

5. Solve the least squares problem

$$\theta_k \in \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \ \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \theta^\top \phi(s_i, a_i)\right)^2$$

→ standard linear regression problem with design matrix and targets

$$X = \begin{pmatrix} \phi(s_1, a_1)^\top \\ \phi(s_2, a_2)^\top \\ \dots \\ \phi(s_n, a_n)^\top \end{pmatrix} \in \mathbb{R}^{n \times d} \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} \in \mathbb{R}^d$$

whose solution is

$$\theta_k = \left(X^\top X\right)^{-1} X^\top Y.$$

Linear Fitted-Q : Error bound

Theorem

Linear FQI with a space $\mathcal{F}$ of d features, with n samples drawn from ρ at each iteration, returns a policy π_K after K iterations which satisfies, w.p. larger than $1-\delta$,

$$\|Q^\star - Q^{\pi_K}\|_\mu \leq \frac{2\gamma}{(1-\gamma)^2} C_{\mu,\rho} \left[\sup_{g\in\mathcal{F}} \inf_{f\in\mathcal{F}} \|T^\star g - f\|_\rho + O\left(\sqrt{\frac{d \log(n/\delta)}{\omega n}} \right) \right] + O\left(\frac{\gamma^K}{(1-\gamma)^2} \right).$$

see, e.g. [Munos and Szepesvári, 2008]

Outline

A semi-gradient extension of Q-Learning

Let's try to find θ minimizing

$$\begin{aligned}\mathtt{MSE}(\theta) &= \mathbb{E}_\nu\left[\left(Q^\star(s,a) - Q_\theta(s,a)\right)^2\right] \\ \nabla_\theta \mathtt{MSE}(\theta) &= -2\mathbb{E}_\nu\left[\left(Q^\star(s,a) - Q_\theta(s,a)\right)\nabla_\theta Q_\theta(s,a)\right]\end{aligned}$$

A semi-gradient extension of Q-Learning

Let's try to find θ minimizing

$$\begin{aligned} \texttt{MSE}(\theta) &= \mathbb{E}_\nu \left[(Q^\star(s,a) - Q_\theta(s,a))^2 \right] \\ \nabla_\theta \texttt{MSE}(\theta) &= -2\mathbb{E}_\nu \left[(Q^\star(s,a) - Q_\theta(s,a)) \nabla_\theta Q_\theta(s,a) \right] \end{aligned}$$

➜ gradient descent :

$$\theta \leftarrow \theta + \alpha \mathbb{E}_\nu \left[(Q^\star(s,a) - Q_\theta(s,a)) \nabla_\theta Q_\theta(s,a) \right]$$

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➜ gradient descent :

$$\theta \leftarrow \theta + \alpha\mathbb{E}_\nu\left[\left(Q^\star(s,a) - Q_\theta(s,a)\right)\nabla_\theta Q_\theta(s,a)\right]$$

➜ stochastic gradient descent : if $(s_t, a_t) \sim \nu$,

$$\theta \leftarrow \theta + \alpha\left(Q^\star(s_t,a_t) - Q_\theta(s_t,a_t)\right)\nabla_\theta Q_\theta(s_t,a_t)$$

A semi-gradient extension of Q-Learning

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$$\theta \leftarrow \theta + \alpha\left(Q^\star(s_t,a_t) - Q_\theta(s_t,a_t)\right)\nabla_\theta Q_\theta(s_t,a_t)$$

➜ bootstrapping : given a transition (s_t, a_t, r_t, s_{t+1}),

$$\theta \leftarrow \theta + \alpha\left(r_t + \gamma \max_b Q_\theta(s_{t+1}, b) - Q_\theta(s_t,a_t)\right)\nabla_\theta Q_\theta(s_t,a_t)$$

A semi-gradient extension of Q-Learning

Let's try to find θ minimizing

$$\begin{aligned}\mathtt{MSE}(\theta) &= \mathbb{E}_\nu\left[\left(Q^\star(s,a) - Q_\theta(s,a)\right)^2\right] \\ \nabla_\theta \mathtt{MSE}(\theta) &= -2\mathbb{E}_\nu\left[\left(Q^\star(s,a) - Q_\theta(s,a)\right)\nabla_\theta Q_\theta(s,a)\right]\end{aligned}$$

Q-Learning update with function approximation

Given a Q-value $Q_\theta(s,a)$, this **semi-gradient** update is

$$\begin{cases} \delta_t = r_t + \gamma \max_b Q_{\theta_{t-1}}(s_{t+1}, b) - Q_{\theta_{t-1}}(s_t, a_t) \\ \theta_t = \theta_{t-1} + \alpha_t \delta_t \nabla_\theta Q_{\theta_{t-1}}(s_t, a_t) \end{cases}$$

➜ one recovers Q-Learning in the tabular case

Negative results

	TD(0)	LSPI	Fitted-Q	Q-Learning
Linear functions	✓	✓	✓	✗
Non-linear functions	✗	✗	(✓)	✗

- TD(0) is known to diverge with non-linear function approximation
- Q-Learning can already diverge with linear function approximation...

(see examples in [Sutton and Barto, 2018])

Tricks to make Q-Learning work

Q-Learning update with function approximation

$$\begin{cases} \delta_t = r_t + \gamma \max_b Q_{\theta_{t-1}}(s_{t+1}, b) - Q_{\theta_{t-1}}(s_t, a_t) \\ \theta_t = \theta_{t-1} + \alpha_t \delta_t \nabla_\theta Q_{\theta_{t-1}}(s_t, a_t) \end{cases}$$

Alternative view : in each step t, perform one SGD step on

$$L(\theta) = \mathbb{E}_{\substack{(s,a)\sim\rho \\ (r,s')\sim\texttt{step}(s,a)}} \left[\left(r + \gamma \max_b Q_{\theta_{t-1}}(s', b) - Q_\theta(s, a) \right)^2 \right]$$

where ρ is the current behavior policy.

Tricks to make Q-Learning work

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Three tricks : (e.g. [Mnih et al., 2015, Hessel et al., 2018])

- ➜ experience replay : rely on past transisions instead of the current one
- ➜ mini-batches : rely on more than one transition
- ➜ two learning scales : do not update the target network in every round

Tricks to make Q-Learning work

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- ➜ experience replay : rely on past transisions instead of the current one
- ➜ mini-batches : rely on more than one transition
- ➜ two learning scales : do not update the target network in every round

Deep Q Networks

Input : number of iterations T,
minimatch size B, update frequency for the target network N,
exploration sequence (ε_t), stepsize (α_t)
Initialize : replay buffer $\mathcal{D} \leftarrow \{\}$, first state s_1,
online network parameter θ, target network parameter $\theta_{-} \leftarrow \theta$

for $t = 1, \ldots, T$ **do**
  $a_t = \mathrm{argmax}_a Q_\theta(s_t, a)$ w.p. $1 - \varepsilon_t$, random action w.p. ε_t
  Perform transition $(r_t, s_{t+1}) = \texttt{step}(s_t, a_t)$
  Add transition to the replay buffer $\mathcal{D} \leftarrow \mathcal{D} \cup \{(s_t, a_t, r_t, s_{t+1})\}$
  Draw a minibatch $\mathcal{B}$ of size B uniformly from $\mathcal{D}$
  Perform one step of online optimization on the loss function

$$L(\theta) = \sum_{(s,a,r,s') \in \mathcal{B}} \left(r + \gamma \max_b Q_{\theta^-}(s', b) - Q_\theta(s, a) \right)^2$$

  e.g. $\theta \leftarrow \theta - \alpha_t \nabla_\theta L(\theta)$
  every N time steps, $\theta^- \leftarrow \theta$
end
Return: Q_θ

Results on Atari Games

DQN was proposed in combination with

- a well chosen pre-processing of the state
- an optimized architecture for the Deep Neural Network used for the approximator

that reaches super-human level performance on Atari games.

[video]

Summary

In this class, we mostly saw how to scale up reinforcement learning with Value-based methods :

- Fitted-Q Iteration
- Deep Q Networks

In the sequel, we will see :

- Policy-based methods (based on direct search over a policy space)
- Actor-critic methods (using both a policy and a value),

whose performance can also be "boosted" with Deep Learning.

We will also discuss the exploration issue : can we go beyond ε-greedy ?
(starting with very simple MDPs : multi-armed bandits)

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