Brownian Motion

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Abstract Our goal is to formalize Brownian motions (or $\mathbb{R}^d$-valued Gaussian processes) in some generality using `Mathlib`.

Outline There are three main parts to this formalization:

- develop the theory of Gaussian distributions,
- build a projective family of Gaussian distributions and define its projective limit by the Kolmogorov extension theorem,
- prove the Kolmogorov-Chentsov continuity theorem.

Notation T denotes an index set (for a stochastic process).
Ω is a measurable space.

Chapter 1

Auxiliary results

`Mathlib` contains a definition of a product measure indexed by a finite set, but does not have lemmas about integrals against such a measure.

Chapter 2

Characteristic functions

Definition 2.1 (Characteristic function)**.** The characteristic function of a measure μ on a normed space E is the function $E^* \to \mathbb{C}$ defined by

$$\hat{\mu}(L) = \int_E e^{iL(x)} \, d\mu(x) \ .$$

Theorem 2.2. *In a separable Banach space, if two finite measures have same characteristic function, they are equal.*

Proof. □

Definition 2.3 (Characteristic function)**.** The characteristic function of a measure μ on an inner product space E is the function $E \to \mathbb{C}$ defined by

$$\hat{\mu}(t) = \int_E e^{i\langle t, x\rangle} \, d\mu(x) \ .$$

This is equal to the normed space version of the characteristic function applied to the linear map $x \mapsto \langle t, x\rangle$.

Theorem 2.4. *In a separable Hilbert space, if two finite measures have same characteristic function, they are equal.*

Proof. □

Lemma 2.5. *Let μ be a measure on F and let $L \in F^*$. Then*

$$\widehat{L_* \mu}(x) = \hat{\mu}(x \cdot L) \ .$$

Proof. □

Chapter 3

Gaussian distributions

3.1 Gaussian measures

3.1.1 Real Gaussian measures

Definition 3.1 (Real Gaussian measure).

Lemma 3.2. *The characteristic function of a real Gaussian measure with mean* μ *and variance* σ^2 *is given by* $x \mapsto \exp\left(i\mu x - \frac{\sigma^2 x^2}{2}\right)$.

Proof. □

3.1.2 Gaussian measures on a Banach space

That kind of generality is not needed for this project, but we happen to have results about Gaussian measures on a Banach space in Mathlib, so we will use them.

Let F be a separable Banach space.

Definition 3.3 (Gaussian measure). A measure μ on F is Gaussian if for every continuous linear form $L \in F^*$, the pushforward measure $L_*\mu$ is a Gaussian measure on $\mathbb{R}$.

Lemma 3.4. *A Gaussian measure is a probability measure.*

Proof. □

Definition 3.5 (Centered measure). A measure μ on F is centered if for every continuous linear form $L \in F^*$, $\mu[L] = 0$.

Theorem 3.6. *A finite measure* μ *on* F *is Gaussian if and only if for every continuous linear form* $L \in F^*$*, the characteristic function of* μ *at* L *is*

$$\hat{\mu}(L) = \exp\left(i\mu[L] - \mathbb{V}_\mu[L]/2\right) ,$$

in which $\mathbb{V}_\mu[L]$ *is the variance of* L *with respect to* μ.

Proof.

□

Theorem 3.7. *Let μ be a finite measure on F such that $\mu \times \mu$ is invariant under the rotation of angle $-\frac{\pi}{4}$. Then there exists $C > 0$ such that the function $x \mapsto \exp(C\|x\|^2)$ is integrable with respect to μ.*

Proof. □

Theorem 3.8 (Fernique's theorem)**.** *For a Gaussian measure, there exists $C > 0$ such that the function $x \mapsto \exp(C\|x\|^2)$ is integrable.*

Proof.

□

Lemma 3.9. *A Gaussian measure μ has finite moments of all orders. In particular, there is a well defined mean $m_\mu := \mu[\mathrm{id}]$, and for all $L \in F^*$, $\mu[L] = L(m_\mu)$.*

Proof.

□

Definition 3.10 (Covariance)**.** The covariance bilinear form of a measure μ with finite second moment is the continuous bilinear form $C_\mu : F^* \times F^* \to \mathbb{R}$ with

$$C_\mu(L_1, L_2) = \int_x (L_1(x) - L_1(m_\mu))(L_2(x) - L_2(m_\mu))\, d\mu(x) .$$

A Gaussian measure has finite second moment by Lemma 3.9, hence its covariance bilinear form is well defined.

Lemma 3.11. *For μ a measure on F with finite second moment and $L \in F^*$, $C_\mu(L, L) = \mathbb{V}_\mu[L]$.*

Proof. □

Transformations of Gaussian measures

Lemma 3.12. *Let F, G be two Banach spaces, let μ be a Gaussian measure on F and let $T : F \to G$ be a continuous linear map. Then $T_*\mu$ is a Gaussian measure on G.*

Proof. □

Corollary 3.13. *Let μ be a Gaussian measure on F and let $c \in F$. Then the measure μ translated by c (the map of μ by $x \mapsto x + c$) is a Gaussian measure on F.*

Proof.

□

Lemma 3.14. *The convolution of two Gaussian measures is a Gaussian measure.*

Proof. □

3.1.3 Gaussian measures on a finite dimensional Hilbert space

Lemma 3.15. *A finite measure μ on a separable Hilbert space E is Gaussian if and only if for every $t \in E$, the characteristic function of μ at t is*

$$\hat{\mu}(t) = \exp\left(i\mu[\langle t, \cdot\rangle] - \mathbb{V}_\mu[\langle t, \cdot\rangle]/2\right) .$$

Proof. By Theorem 3.6, μ is Gaussian iff for every continuous linear form $L \in E^*$, the characteristic function of μ at L is

$$\hat{\mu}(L) = \exp\left(i\mu[L] - \mathbb{V}_\mu[L]/2\right) .$$

Every continuous linear form $L \in E^*$ can be written as $L(x) = \langle t, x\rangle$ for some $t \in E$, hence we have that μ is Gaussian iff for every $t \in E$,

$$\hat{\mu}(t) = \exp\left(i\mu[\langle t, \cdot\rangle] - \mathbb{V}_\mu[\langle t, \cdot\rangle]/2\right) .$$

□

Let E be a finite dimensional Hilbert space. We denote by $\langle \cdot, \cdot\rangle$ the inner product on E and by $\|\cdot\|$ the associated norm.

Definition 3.16 (Covariance matrix)**.** The covariance matrix of a Gaussian measure μ on E is the positive semidefinite matrix Σ_μ with

$$\Sigma_\mu = \mu[(x - m_\mu)(x - m_\mu)^\top] .$$

Lemma 3.17. *The characteristic function of a Gaussian measure μ on E is given by*

$$\hat{\mu}(t) = \exp\left(i\langle t, m_\mu\rangle - \frac{1}{2}\langle t, \Sigma_\mu t\rangle\right) .$$

Proof. By Lemma 3.15, for every $t \in E$,

$$\hat{\mu}(t) = \exp\left(i\mu[\langle t, \cdot\rangle] - \mathbb{V}_\mu[\langle t, \cdot\rangle]/2\right) .$$

By Lemma 3.9, μ has finite first moment and $\mu[\langle t, \cdot\rangle] = \langle t, m_\mu\rangle$.

TODO: the second moment is also finite and we can get to the covariance matrix. □

Lemma 3.18. *A finite measure μ on E is Gaussian if and only if there exists $m \in E$ and Σ positive semidefinite such that for all $t \in E$, the characteristic function of μ at t is*

$$\hat{\mu}(t) = \exp\left(i\langle t, m\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right) ,$$

If that's the case, then $m = m_\mu$ and $\Sigma = \Sigma_\mu$.

Note that this lemma does not say that there exists a Gaussian measure for any such m and Σ. We will prove that later.

Proof. Lemma 3.17 states that the characteristic function of a Gaussian measure has the wanted form.

Suppose now that there exists $m \in E$ and Σ positive semidefinite such that for all $t \in E$, $\hat{\mu}(t) = \exp\left(i\langle t, m\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right)$.

We need to show that for all $L \in E^*$, $L_*\mu$ is a Gaussian measure on $\mathbb{R}$. Such an L can be written as $\langle u, \cdot \rangle$ for some $u \in E$. Let then $u \in E$. We compute the characteristic function of $\langle u, \cdot \rangle_* \mu$ at $x \in \mathbb{R}$ with Lemma 2.5:

$$\begin{aligned} \widehat{\langle u, \cdot \rangle_* \mu}(x) &= \hat{\mu}(x \cdot u) \\ &= \exp\left(ix\langle u, m\rangle - \frac{1}{2}x^2 \langle u, \Sigma u\rangle\right) . \end{aligned}$$

This is the characteristic function of a Gaussian measure on $\mathbb{R}$ with mean $\langle u, m\rangle$ and variance $\langle u, \Sigma u\rangle$. By Theorem 2.4, $\langle u, \cdot \rangle_* \mu$ is Gaussian, hence μ is Gaussian. □

Definition 3.19 (Standard Gaussian measure)**.** Let $(e_1, \dots, e_d)$ be an orthonormal basis of E and let $\mu_1, \dots, \mu_d$ be independent standard Gaussian measures on $\mathbb{R}$. The standard Gaussian measure on E is the pushforward measure of the product measure $\mu_1 \times \dots \times \mu_d$ by the map $x \mapsto \sum_{i=1}^d x_i \cdot e_i$.

Lemma 3.20. *The standard Gaussian measure on E is centered, i.e., $\mu[L] = 0$ for every $L \in E^*$.*

Proof. □

Lemma 3.21. *THe standard Gaussian measure is a probability measure.*

Proof. □

Lemma 3.22. *The characteristic function of the standard Gaussian measure on E is given by*

$$\hat{\mu}(t) = \exp\left(-\frac{1}{2}\|t\|^2\right) .$$

Proof.

□

Lemma 3.23. *The standard Gaussian measure on E is a Gaussian measure.*

Proof. Since the standard Gaussian is a probability measure (hence finite), we can apply Lemma 3.18 that states that it suffices to show that the characteristic function has a particular form. That form is given by Lemma 3.22. □

Definition 3.24 (Multivariate Gaussian)**.** The multivariate Gaussian measure on $\mathbb{R}^d$ with mean $m \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$, with Σ positive semidefinite, is the pushforward measure of the standard Gaussian measure on $\mathbb{R}^d$ by the map $x \mapsto m + \Sigma^{1/2}x$. We denote this measure by $\mathcal{N}(m, \Sigma)$.

Theorem 3.25. *The characteristic function of a multivariate Gaussian measure $\mathcal{N}(m, \Sigma)$ is given by*

$$\hat{\mu}(t) = \exp\left(i\langle m, t\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right) .$$

Proof.

□

Lemma 3.26. *A multivariate Gaussian measure is a Gaussian measure.*

Proof. Apply Lemma 3.18 that states that it suffices to show that the characteristic function has a particular form. That form is given by Theorem 3.25. □

3.2 Gaussian processes

Definition 3.27 (Gaussian process)**.** A process $X : T \to \Omega \to E$ is Gaussian if for every finite subset $t_1, \dots, t_n \in T$, the random vector $(X_{t_1}, \dots, X_{t_n})$ has a Gaussian distribution.

Chapter 4

Projective family of the Brownian motion

4.1 Kolmogorov extension theorem

This theorem has been formalized in the repository kolmogorov_extension4.

Definition 4.1 (Projective family)**.** A family of measures P indexed by finite sets of T is projective if, for finite sets $J \subseteq I$, the projection from E^I to E^J maps P_I to P_J.

Definition 4.2 (Projective limit)**.** A measure μ on E^T is the projective limit of a projective family of measures P indexed by finite sets of T if, for every finite set $I \subseteq T$, the projection from E^T to E^I maps μ to P_I.

Theorem 4.3 (Kolmogorov extension theorem)**.** *Let $\mathcal{X}$ be a Polish space, equipped with the Borel σ-algebra, and let T be an index set. Let P be a projective family of finite measures on $\mathcal{X}$. Then the projective limit μ of P exists, is unique, and is a finite measure on $\mathcal{X}^T$. Moreover, if P_I is a probability measure for every finite set $I \subseteq T$, then μ is a probability measure.*

Proof. □

4.2 Projective family of Gaussian measures

We build a projective family of Gaussian measures indexed by $\mathbb{R}_+$. In order to do so, we need to define specific Gaussian measures on finite index sets $\{t_1, \dots, t_n\}$. We want to build a multivariate Gaussian measure on $\mathbb{R}^n$ with mean 0 and covariance matrix $C_{ij} = \min(t_i, t_j)$ for $1 \le i, j \le n$.

First method: Gaussian increments In this method, we build the Gaussian measure by adding independent Gaussian increments.

Definition 4.4. (Gaussian increment) For $v \ge 0$, the map from $\mathbb{R}$ to the probability measures on $\mathbb{R}$ defined by $x \mapsto \mathcal{N}(x, v)$ is a Markov kernel. We call that kernel the *Gaussian increment* with variance v and denote it by κ_v^G.

TODO: perhaps the equality $\mathcal{N}(x, v) = \delta_x * \mathcal{N}(0, v)$ is useful to show that it is a kernel?

Definition 4.5. Let $0 \le t_1 \le \dots \le t_n$ be non-negative reals. Let μ_0 be the real Gaussian distribution $\mathcal{N}(0, t_1)$. For $i \in \{1, \dots, n-1\}$, let κ_i be the Markov kernel from $\mathbb{R}$ to $\mathbb{R}$ defined by $\kappa_i(x) = \mathcal{N}(x, t_{i+1} - t_i)$ (the Gaussian increment $\kappa^G_{t_{i+1}-t_i}$). Let $P_{t_1,\dots,t_n}$ be the measure on $\mathbb{R}^n$ defined by $\mu_0 \otimes \kappa_1 \otimes \dots \otimes \kappa_{n-1}$.

TODO: explain the notation $\otimes$ in the lemma above: κ_{n-1} takes the value at $n-1$ only to produce the distribution at n.

Lemma 4.6. *$P_{t_1,\dots,t_n}$ is a Gaussian measure on $\mathbb{R}^n$ with mean 0 and covariance matrix $C_{ij} = \min(t_i, t_j)$ for $1 \le i, j \le n$.*

Proof. □

Second method: covariance matrix In this method, we prove that the matrix $C_{ij} = \min(t_i, t_j)$ is positive semidefinite, which means that there exists a Gaussian distribution with mean 0 and covariance matrix C.

Lemma 4.7. *For $I = \{t_1, \dots, t_n\}$ a finite subset of $\mathbb{R}_+$, let $C \in \mathbb{R}^{n\times n}$ be the matrix $C_{ij} = \min(t_i, t_j)$ for $1 \le i, j \le n$. Then C is positive semidefinite.*

Proof. □

Definition of the projective family and extension

Definition 4.8 (Projective family of the Brownian motion)**.** For $I = \{t_1, \dots, t_n\}$ a finite subset of $\mathbb{R}_+$, let P^B_I be the multivariate Gaussian measure on $\mathbb{R}^n$ with mean 0 and covariance matrix $C_{ij} = \min(t_i, t_j)$ for $1 \le i, j \le n$. We call the family of measures P^B_I the *projective family of the Brownian motion.*

Lemma 4.9. *The projective family of the Brownian motion is a projective family of measures.*

Proof. □

Definition 4.10. We denote by TODO the projective limit of the projective family of the Brownian motion given by Theorem 4.3. This is a probability measure on $\mathbb{R}^{\mathbb{R}_+}$.

Chapter 5

Kolmogorov-Chentsov Theorem

5.1 Covers

Let (E, d_E) be a pseudometric space.

Definition 5.1 (ε-cover)**.** A set $C \subseteq E$ is an ε-cover of a set $A \subseteq E$ if for every $x \in A$, there exists $y \in C$ such that $d_E(x, y) < \varepsilon$.

Definition 5.2 (External covering number)**.** The external covering number of a set $A \subseteq E$ for $\varepsilon \geq 0$ is the smallest cardinality of an ε-cover of A. Denote it by $N_\varepsilon^{ext}(A)$.

Definition 5.3 (Internal covering number)**.** The internal covering number of a set $A \subseteq E$ for $\varepsilon \geq 0$ is the smallest cardinality of an ε-cover of A which is a subset of A. Denote it by $N_\varepsilon^{int}(A)$.

Lemma 5.4. *$N_\varepsilon^{ext}(A) \leq N_\varepsilon^{int}(A)$.*

Proof. □

Lemma 5.5. *For $I = [0, 1] \subseteq \mathbb{R}$, $N_\varepsilon^{int}(I) \leq 1/\varepsilon$.*

Proof. □

5.2 Chaining

Lemma 5.6. *Let (I, d_I) and (E, d_E) be metric spaces, and $f : I \to E$. Moreover, let $J \subseteq I$ be finite, $a, b, c \in \mathbb{R}_+$ with $a \geq 1$ and $n \in \{1, 2, ...\}$ such that $|J| \leq ba^n$. Then, there is $K \subseteq J^2$ such that*

$$|K| \leq a|J|, \tag{5.1}$$

$$(s, t) \in K \implies d_I(s, t) \leq cn, \tag{5.2}$$

$$\sup_{s,t \in J, d_I(s,t) \leq c} |f(t) - f(s)| \leq 2 \sup_{(s,t) \in K} |f(s) - f(t)|. \tag{5.3}$$

Proof. □

5.3 Kolmogorov-Chentsov Theorem

Theorem 5.7 (Continuous version; Kolmogorov, Chentsov)**.** *Let* (I, d_I) *be a compact metric space. Suppose that there is* $c_1 > 0$ *and* $d \in \mathbb{N}$ *such that for all* $\varepsilon > 0$ *small enough,* $N_\varepsilon^{int}(I) \leq c_1 \varepsilon^{-d}$*. Assume that* $X = (X_t)_{t \in I}$ *is an* E*-valued stochastic process and there are* $\alpha, \beta, c_2 > 0$ *with*

$$\mathbb{E}[d_E(X_s, X_t)^\alpha] \leq c_2 d_I(s,t)^{d+\beta}, \qquad s, t \in I \; .$$

Then, there exists a version $Y = (Y_t)_{t \in I}$ *of* X *such that, for some random variables* $H > 0$ *and* $K < \infty$*,*

$$\mathbb{P}\Big(\sup_{s \neq t, d_I(s,t) \leq H} d_E(Y_s, Y_t)/d_I(s,t)^\gamma \leq K \Big) = 1 \; ,$$

for every $\gamma \in (0, \beta/\alpha)$*. In particular,* Y *almost surely is locally Hölder of all orders* $\gamma \in (0, \beta/\alpha)$*, and has continuous paths.*

Proof.

□

5.4 Brownian motion